SEMI-CLASSICAL ASYMPTOTICS FOR THE COUNTING FUNCTIONS AND RIESZ MEANS OF PAULI AND DIRAC OPERATORS WITH LARGE MAGNETIC FIELDS

A. A. BALINSKY, W. D. EVANS, AND ROGER T. LEWIS

Abstract. We study the asymptotic behavior, as Planck's constant $\hbar \to 0$, of the number of discrete eigenvalues and the Riesz means of Pauli and Dirac operators with a magnetic field $\mu\mathbf{B}(x)$ and an electric field. The magnetic field strength μ is allowed to tend to infinity as $\hbar \to 0$. Two main types of results are established: in the first $\mu\hbar \le$ *constant* as $\hbar \to 0$, with magnetic fields of arbitrary direction; the second results are uniform with respect to $\mu \ge 0$ but the magnetic fields have constant direction. The results on the Pauli operator complement recent work of Sobolev.

1. Introduction

The Dirac and Pauli operators $\mathbb{D}_V, \mathbb{P}_W$, which are the objects of study in this paper, are defined as follows :

$$\begin{aligned}
\mathbb{D}_V \equiv \mathbb{D}_V(\mathbf{B}) &:= \boldsymbol{\alpha}\cdot(\frac{\hbar}{i}\nabla - \mu\boldsymbol{a}) + \beta + V \\
&\equiv \sum_{k=1}^{3}\boldsymbol{\alpha}_k(-i\hbar\partial_k - \mu\boldsymbol{a}_k) + \beta + V \qquad (1)\\
\mathbb{P}_W \equiv \mathbb{P}_W(\mathbf{B}) &:= [\boldsymbol{\sigma}\cdot(-i\hbar\nabla - \mu\boldsymbol{a})]^2 + W \\
&\equiv [\sum_{k=1}^{3}\boldsymbol{\sigma}_k(-i\hbar\partial_k - \mu\boldsymbol{a}_k)]^2 + W \\
&= H_0(\mathbf{B}) - \mu\hbar\boldsymbol{\sigma}\cdot\mathbf{B} + W, \qquad (2)
\end{aligned}$$

where $H_0(\mathbf{B}) = (-i\hbar\nabla - \mu\mathbf{a})^2$ is the Schrödinger operator with magnetic field $\mathbf{B}$ and

- $\boldsymbol{a} = (a_1, a_2, a_3)$ is a magnetic vector potential with magnetic field $\mathbf{B} := \nabla\times\boldsymbol{a}$;
- $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the triple of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

- $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ where α_j, $j = 1, 2, 3$, and β are the Dirac matrices

$$\alpha_k = \begin{pmatrix} 0_2 & \sigma_k \\ \sigma_k & 0_2 \end{pmatrix},\ k = 1, 2, 3, \qquad \beta = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix},$$

where $0_2, I_2$ are the 2×2 zero and unit matrices respectively.

Date: February 7, 2008.
Key words and phrases. Semi-classical asymptotics, Pauli and Dirac operators, counting functions, Riesz means.

Given a domain $\Omega \subseteq \mathbb{R}^3$, self-adjoint realizations $\mathbb{P}_W(\Omega)$ and $\mathbb{D}_V(\Omega)$ are defined by Dirichlet boundary conditions on Ω, these being satisfied in the usual weak sense (see §2). The Pauli operator $\mathbb{P}_W(\Omega)$ acts in $L^2(\Omega) \otimes \mathbb{C}^2 \equiv [L^2(\Omega)]^2$ and the Dirac operator $\mathbb{D}_V(\Omega)$ in $L^2(\Omega) \otimes \mathbb{C}^4 \equiv [L^2(\Omega)]^4$. Under quite general conditions the spectrum of $\mathbb{P}_0$ coincides with $[0, \infty)$, and the perturbation W introduces negative eigenvalues $\lambda_n(\mathbb{P}_W(\Omega))$. The Dirac operator $\mathbb{D}_0$ has typically a spectrum $\mathbb{R} \setminus (-1, 1)$ and V causes eigenvalues $\lambda_n(\mathbb{D}_V(\Omega))$ to appear in the gap $(-1, 1)$. Our concern in this paper is with the Riesz means

$$M_\gamma(\mathbb{P}_W, \Omega) = \sum_n |\lambda_n(\mathbb{P}_W(\Omega))|^\gamma, \quad M_\gamma(\mathbb{D}_V, \Omega) = \sum_n |\lambda_n(\mathbb{D}_V(\Omega))|^\gamma \tag{3}$$

where $\gamma \geq 0$, and, in particular, the counting functions given by

$$N(\mathbb{P}_W, \Omega) = M_0(\mathbb{P}_W, \Omega), \quad N(\mathbb{D}_V, \Omega) = M_0(\mathbb{D}_V, \Omega). \tag{4}$$

In [23], Sobolev investigated the natural quasi-classical formula

$$M_\gamma(\mathbb{P}_W, \Omega) \sim \hbar^{-3} \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|, W, \Omega), \hbar \to 0, \tag{5}$$

with the "magnetic" Weyl coefficient

$$\mathfrak{B}_\gamma(b, v, \Omega) := \beta_\gamma \int_\Omega b(x) \left[v_-(x)^{\gamma+\frac{1}{2}} + 2 \sum_{k=1}^\infty [2kb(x) + v(x)]_-^{\gamma+\frac{1}{2}} \right] dx, \tag{6}$$

where

$$\beta_\gamma = \frac{1}{4\pi^2} \int_0^1 t^\gamma (1-t)^{-1/2} dt.$$

Under weak regularity conditions on $\mathbf{B}$ and W, Sobolev established two main results:

(R_1): for $\mu\hbar \leq$ constant, (5) is satisfied for $\gamma \geq 1$;
(R_2): if $\mathbf{B}$ has constant direction, (5) is satisfied uniformly in $\mu \geq 0$ for $\gamma > 1/2$.

Earlier results of this type were proved for homogeneous (i.e. constant) fields by Lieb, Solovej, and Yngvason [15], [16], and for non-homogeneous fields by Erdős and Solovej [7], [8], with $\mu\hbar^3 \to 0$. Our objective in this paper is to investigate the validity of (R_1) and (R_2) for all $\gamma \geq 0$. We prove in Theorem 1 that if $W, |\mathbf{B}| \in L^{3/2}(\Omega)$ in (R_1), then (5) is satisfied for all $\gamma \geq 0$. This extends a result in [9] in which a Weyl asymptotic formula is established for the case $\mu\hbar \sim 0$ as $\hbar \to 0$. If $|\mathbf{B}|$ does not belong to $L^{3/2}(\Omega)$ in (R_1), we obtain the result for $N(\mathbb{P}_W + \lambda, \Omega)$, the number of eigenvalues of $\mathbb{P}_W(\Omega)$ less than $-\lambda < 0$. Our main result in problem (R_2) is also of this form. This is the best that can be expected in general for $\gamma = 0$ in view of the absence of an inequality of Cwikel, Lieb, Rozenblum type for the number of negative eigenvalues; such an inequality is available when $|\mathbf{B}| \in L^{3/2}(\Omega)$. We also investigate circumstances in which (5) holds for $\gamma > 0$ when $\mathbf{B}$ has a constant direction.

In [18] the leading term in the asymptotic value of $N(\mathbb{P}_W + \lambda, \mathbb{R}^3)$ is determined for $\lambda > 0, \hbar = 1, \mathbf{B}$ of constant direction, and $\mu \to \infty$. Specifically, with $\mathbf{B}(\mathbf{x}) = (0, 0, B(x)), \mathbf{x} = (x, x_3), x \in \mathbb{R}^2$, where $B(x)$ is bounded above and away from zero and has a bounded gradient,

$$\lim_{\mu\to\infty} \mu^{-1} N(\mathbb{P}_W + \lambda, \mathbb{R}^3) = \mathcal{D}(\lambda) := \frac{1}{2\pi} \int_{\mathbb{R}^2} N(X_{W+\lambda}(x), \mathbb{R}) B(x) dx, \tag{7}$$

where, for any fixed $x \in \mathbb{R}^2, X_W(x)$ is a self-adjoint realization of $-\frac{d^2}{dx_3^2} + W(x,\cdot)$ in $L^2(\mathbb{R})$ with essential spectrum $[0,\infty)$. From many other results on the spectral asymptotics of $N(\mathbb{P}_W,\Omega)$, we mention in particular that of Iwatsuka and Tamura in [11] for $\hbar = \mu = 1$ and $\lambda \to 0$:

$$N(\mathbb{P}_W + \lambda, \mathbb{R}^3) \sim \mathcal{D}(\lambda). \tag{8}$$

Results for the two-dimensional problems are also obtained by the authors cited above, and our techniques also apply in this case. Furthermore, we derive analogous results for the Dirac operator with magnetic field in both cases (R_1) and (R_2) above.

Our strategy is based on that in [23] which in turn was inspired by ideas from [3]. For the case $\mu\hbar \leq$constant, the technique we use involves a tesselation of $\mathbb{R}^3$ by cubes Q, the derivation of two-sided estimates for $N(\mathbb{P}_W,Q)$, with constant W and $\mathbf{B}$, in terms of the explicit values of the eigenvalues (Landau levels) of the operator realization of $\mathbb{P}_W$ on a torus, and subsequently the localization of $\mathbb{P}_W(\Omega)$ in terms of the $\mathbb{P}_W(Q)$ and the application of either the Cwikel, Lieb, Rozenblum (CLR) inequality for the magnetic Schrödinger operator (in the case in which there is known to be a finite number of negative eigenvalues in problem (R_1)), or an inequality derived from a Lieb-Thirring inequality established by Shen in [20] for the trace of the negative eigenvalues. For problem (R_2) this method is too crude. Sobolev in [23] used the spectral properties of $\mathbb{P}_W$ on a torus with an arbitrary periodic magnetic field $\mathbf{B}$ having integer flux, and, in particular, that zero is an eigenvalue of the operator with multiplicity equal to the flux of $\mathbf{B}$. Another important ingredient in [23] in this case is a Lieb-Thirring inequality established in [22] in which $|\mu\mathbf{B}|$ has the proper (linear) scale; for a discussion of the significance of this see the paper of Erdős and Solovej [7], §1. We use a similar estimate derived by Shen in [20] for $N(\mathbb{P}_W + \lambda, \mathbb{R}^3)$ with $\lambda > 0$. This grows like $1/\sqrt{\lambda}$ as $\lambda \to 0$, and implies (5) only when $\gamma > 1/2$, as already established in [23] with similar assumptions. We also present a result for a cylindrical domain Ω in which $N(\mathbb{P}_W + \lambda, \Omega)$ grows logarithmically, which implies that $M_\gamma(\mathbb{P}_W,\Omega)$ is finite for all $\gamma > 0$ (see Proposition 5).

Throughout the paper $(\cdot,\cdot)_\Omega$, $\|\cdot\|_\Omega$ are used to denote the usual inner-product and norm in each of the Hilbert spaces $L^2(\Omega)$, $[L^2(\Omega)]^2$ and $[L^2(\Omega)]^4$; the precise space will be clear from the context. We adopt the convention that inner-products are linear in the second argument and conjugate linear in the first. We denote the identity matrix on $\mathbb{C}^4 = \mathbb{C}^2 \bigotimes \mathbb{C}^2$ by $\mathbb{I}_2$. We denote points in $\mathbb{R}^3$ as $\mathbf{x} = (x,x_3), x \in \mathbb{R}^2, x_3 \in \mathbb{R}$. We shall write $A \underset{\sim}{<} B$, or $B \underset{\sim}{>} A$, to mean that $|A| \leq CB$ for some positive constant C, and $f(x) = O(A)$ will mean that $f(x)$ is bounded by A.

2. Preliminaries

To begin, we need to define the operators precisely and locate their essential spectra. The Pauli operator $\mathbb{P}_W$ will be defined as the form sum of $\mathbb{P}_0$ and the operator of multiplication by W, the conditions assumed on W being sufficient for the associated sesquilinear form to be lower semi-bounded. For the Dirac operator we require the restriction of $\mathbb{D}_V$ to $C_0^\infty(\Omega)$ to be essentially self-adjoint, in which case ${\mathbb{D}_V}^2$ is the self-adjoint operator associated with the form $\|\mathbb{D}_V\phi\|_\Omega^2$. In §3 below, our requirements in Theorem 1 are met by the following results from [9]. Define, for $m = 2,4$,

$$\mathcal{H}_a^1 \equiv \mathcal{H}_a^1(\mathbb{R}^3) := \{u : u, ((\hbar/i)\partial_\nu - \mu a_\nu)u \in [L^2(\mathbb{R}^3)]^m, \nu = 1,2,3,\}$$

in which the derivatives are defined in the distributional sense. If $\mathbf{a} \in [L^2_{loc}]^3$, then $[C_0^\infty(\mathbb{R}^3)]^m$ is dense in $\mathcal{H}^1_a$ (see Kato [12], Simon [21], and Leinfelder and Simader [17]). Also, the diamagnetic inequality

$$|(\hbar/i)\partial_\nu(|u|)| \le |((\hbar/i)\partial_\nu - \mu a_\nu)u|, \qquad \nu = 1,2,3,$$

holds for almost every $x \in \mathbb{R}^3$ and $u \in \mathcal{H}^1_a$ (see Lieb and Loss [13], p.179). As a consequence, $u \mapsto |u|$ maps $\mathcal{H}^1_a$ continuously into the Sobolev space $H^1(\mathbb{R}^3)$, which implies the existence of a continuous embedding

$$\mathcal{H}^1_a \hookrightarrow [L^s(\mathbb{R}^3)]^m, \qquad s \in [2,6]$$

(see Edmunds and Evans [5], Theorem V.3.7). We denote by $\mathcal{H}^1_{a,0}(\Omega)$ the closure of $[C_0^\infty(\Omega)]^m$ in $\mathcal{H}^1_a$. The following propositions follow in a similar way to Lemma 2.1 and Lemma 2.2 in [9] and help to satisfy our needs in Theorem 1.

Proposition 1. *Let $a_\nu \in L^2_{loc}(\Omega)$, $\nu = 1,2,3$, and W, $|\mathbf{B}| \in L^{3/2}(\Omega)$. Then $\mathbb{P}_W = \mathbb{P}_0 + W$ is defined as a form sum with form domain $\mathcal{H}^1_{a,0}(\Omega)$.*

Proposition 2. *Let $\mathbf{a}$, $V(= W)$, $|\mathbf{B}|$ satisfy the conditions of Proposition 1 and suppose that the a_ν are locally Lipschitz on Ω. Then $\mathbb{D}_V$ is essentially self-adjoint on $[C_0^\infty(\Omega)]^4$. If furthermore $V \in L^3(\Omega)$, then $\mathbb{D}_V$ has domain $\mathcal{H}^1_{a,0}(\Omega)$.*

At the end of §2 of [9] conditions, consistent with those in Proposition 2, are given which ensure that $\mathbb{D}_V$ has essential spectrum $\mathbb{R} \setminus (-1,1)$.

The assumptions made in Propositions 1 and 2 are in fact sufficient for there to exist only a finite number of negative eigenvalues of $\mathbb{P}_W(\Omega)$ (and in $(-1,1)$ for $\mathbb{D}_V(\Omega)$). To consider cases in which $|\mathbf{B}|$ is not in $L^{3/2}(\Omega)$ we make use of an estimate given by Shen in [20] for the trace $M_1(\mathbb{P}_W, \mathbb{R}^3)$. The assumptions that this requires are sufficient to ensure that $\mathbb{P}_W$ is defined as a form sum. To be specific, let

$$l_p(\mathbf{x}) := \sup\left\{ l > 0 : l^2 \left(\frac{1}{l^3} \int_{Q(\mathbf{x},l)} |B(\mathbf{y})|^p d\mathbf{y} \right)^{1/p} \le 1 \right\},$$

where $Q(\mathbf{x}, l)$ denotes the cube in $\mathbb{R}^3$ center $\mathbf{x}$ and side l. Define

$$b_p(\mathbf{x}) := \frac{1}{[l_p(\mathbf{x})]^2}. \tag{9}$$

This is Shen's "effective" magnetic field. We recall that the need to replace $|\mathbf{B}|$ by some screened version in Lieb-Thirring inequalities for Pauli operators with non-homogeneous magnetic fields was demonstrated by Erdos in [6], and this had motivated Sobolev in [22] to obtain Lieb-Thirring estimates similar in form to that obtained for constant fields in Lieb, Solovej, and Yngvason [15] but with $|\mathbf{B}|$ replaced by an "effective" magnetic field. Another result of this kind is established in [2]. The novelty of Shen's approach lies in the simple and natural way in which his b_p is constructed. In [20], Remark 1.4, the following result is proved for $\Omega = \mathbb{R}^3$, but the proof is valid in general. It also holds if b_p is Sobolev's effective magnetic field, or that of Bugliaro *et al*.

Proposition 3. *Let $\mathbf{a} \in L^2_{loc}(\Omega, \mathbb{R}^3)$ and suppose that for any $p > 3/2$ and $\gamma \ge 1$, $W_-^{\gamma+3/2}$, $b_p^{3/2} W_-^\gamma \in L^1(\Omega)$. Then $\mathbb{P}_W$ is defined as a form sum with form domain $\mathcal{H}^1_{a,0}$.*

In the case when $\mathbf{B}$ has constant direction, there is another result of Shen [20] which fits our purpose, namely his estimate for the counting function $N(\mathbb{P}_W+\lambda,\mathbb{R}^3)$. Let $\mathbf{B}(\mathbf{x})=(0,0,B(x))$, where $\mathbf{x}=(x,x_3), x\in\mathbb{R}^2$, and define

$$l_p(x):=\sup\left\{l>0: l^2\left(\frac{1}{l^2}\int_{S(x,l)}|B(y)|^p dy\right)^{1/p}\leq 1\right\},$$

where $S(x,l)$ denotes the square in $\mathbb{R}^2$ center x and side l. Define

$$\hat{b}_p(x):=\frac{1}{[l_p(x)]^2}. \tag{10}$$

The following result is proved in [[20], Remark 1.4].

Proposition 4. *Let $\mathbf{a}\in L^2_{loc}(\Omega,\mathbb{R}^3)$ and suppose that for any $p>1$ and $\gamma>1/2$, $W_-^{\gamma+3/2}$, $\hat{b}_pW_-^{\gamma+1/2}\in L^1(\Omega)$. Then, with $\mathbf{B}=(0,0,B)$, $\mathbb{P}_W$ is defined as a form sum with form domain $\mathcal{H}^1_{a,0}$.*

In the rest of this section we present preparatory results for subsequent sections. Our initial assumptions are as follows:

(A_1) Let $Q=\cup_{k=1}^K Q_k$, where $\{Q_k\subseteq\Omega : k=1,\ldots,K\}$ is a finite collection of non-overlapping congruent cubes whose edges are parallel to the coordinate axes and of length r. The set Q is fixed, but the side lengths r and number K will depend on $\hbar$ in due course.

(A_2) Let V_0 and W_0 be piecewise constant functions, taking constant values in each Q_k, and zero outside Q.

(A_3) Assume that $\mathbf{B}$ is continuous. Define $\mathbf{B}^o$ on Q by $\mathbf{B}^o(\mathbf{x})=\mathbf{B}(\mathbf{x}_k)$, $\mathbf{x}\in Q_k$, where $\mathbf{x}_k$ is the center of the cube Q_k, and let $\mathbf{B}^o(\mathbf{x})\equiv 0$ for $\mathbf{x}\notin Q$. Choose[1] gauges $\boldsymbol{a}$ and $\mathring{\mathbf{a}}$ for $\mathbf{B}$ and $\mathbf{B}^o$, respectively, such that for every Q_k

$$\max_{\mathbf{x}\in Q_k}|\boldsymbol{a}(\mathbf{x})-\mathring{\mathbf{a}}(\mathbf{x})|\leq Cr\sigma_r,\qquad \sigma_r:=\max_{|\mathbf{x}-\mathbf{y}|<r}|\mathbf{B}(\mathbf{x})-\mathbf{B}(\mathbf{y})|. \tag{11}$$

Define

$$\mathfrak{B}_\gamma(b,v,\Omega):=\beta_\gamma\int_\Omega b(\mathbf{x})\left[v_-(\mathbf{x})^{\gamma+\frac12}+2\sum_{k=1}^\infty[2kb(\mathbf{x})+v(\mathbf{x})]_-^{\gamma+\frac12}\right]d\mathbf{x} \tag{12}$$

for $\gamma\geq 0$ where

$$\beta_\gamma:=\frac{1}{4\pi^2}\int_0^1(1-t)^{-\frac12}t^\gamma dt.$$

For simplicity, we shall write $\mathfrak{B}(b,v,\Omega)$ for $\mathfrak{B}_0(b,v,\Omega)$ when there is no danger of confusion. We are interested in the asymptotic behavior of the Riesz means

$$\sum_k|\lambda_k|^\gamma,\quad \gamma\geq 0,$$

where each λ_k is a negative eigenvalue of the Pauli operator $\mathbb{P}_W(\Omega)$, or each λ_k is an eigenvalue of the Dirac operator $\mathbb{D}_V$ in $(-1,1)$.

We make frequent use of Proposition 3.2 of Sobolev [23], which follows from results of Colin de Verdiere [3]. It states that there exists a constant C such that for any $\delta\in(0,\frac12)$

$$\hbar^3N(\mathbb{P}_0(\mathbf{B}^o)+\lambda,Q_k)\leq\mathfrak{B}(\mu\hbar|\mathbf{B}^o|,\lambda,Q_k) \tag{13}$$

[1] See the Appendix for details.

and

$$\hbar^3 N(\mathbb{P}_0(\mathbf{B}^o)+\lambda, Q_k) \geq (1-\delta)^3 \mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \lambda + \frac{C\hbar^2}{\delta^2 r^2}, Q_k). \tag{14}$$

Here $\mathbb{P}_0(\mathbf{B}^o)(Q_k)$ is the Dirichlet operator on Q_k, and $N(\mathbb{P}_0(\mathbf{B}_0)+\lambda, Q_k)$ the number of its negative eigenvalues below $-\lambda < 0$.

Lemma 1. *Define* $\boldsymbol{a}^\infty := \boldsymbol{a} - \mathring{\mathbf{a}}$. *For all* $\phi \in [C_0^\infty(Q_k)]^4$ *and* $\theta > 0$

$$\begin{aligned} &|([(\mathbb{D}_{V_0}(\mathbf{B}) - V_0)^2 - 1 + W_0 - (\mathbb{P}_0(\mathbf{B}^o) + W_0)\mathbb{I}_2]\phi, \phi)_{Q_k}| \\ &\leq \theta(\mathbb{P}_0(\mathbf{B}^o)\mathbb{I}_2\phi, \phi)_{Q_k} + \mu^2(1+\tfrac{1}{\theta})(|\boldsymbol{a}^\infty|^2\phi, \phi)_{Q_k}. \end{aligned} \tag{15}$$

Proof. Set $D_{\boldsymbol{a}} := \frac{\hbar}{i}\nabla - \mu\boldsymbol{a}$ and $D_0 = D_{\mathring{\mathbf{a}}}$. We have that

$$\begin{aligned} &([(\mathbb{D}_{V_0} - V_0)^2 - 1 + W_0]\phi, \phi)_{Q_k} \\ &= ([(\boldsymbol{\alpha}\cdot D_{\boldsymbol{a}} + \beta)^2 - 1 + W_0]\phi, \phi)_{Q_k} \\ &= \|[\boldsymbol{\alpha}\cdot(D_0 - \mu\boldsymbol{a}^\infty)\phi\|_{Q_k}^2 + (W_0\phi, \phi)_{Q_k} \\ &= \|\boldsymbol{\alpha}\cdot D_0\phi\|_{Q_k}^2 - 2\mu\mathfrak{Re}[([\boldsymbol{\alpha}\cdot D_0]\phi, [\boldsymbol{\alpha}\cdot\boldsymbol{a}^\infty]\phi)_{Q_k}] + \mu^2\|[\boldsymbol{\alpha}\cdot\boldsymbol{a}^\infty]\phi\|_{Q_k}^2 + (W_0\phi, \phi)_{Q_k} \\ &= (\mathbb{P}_{W_0}(\mathbf{B}^o)\mathbb{I}_2\phi, \phi)_{Q_k} - 2\mu\mathfrak{Re}[([\boldsymbol{\alpha}\cdot D_0]\phi, [\boldsymbol{\alpha}\cdot\boldsymbol{a}^\infty]\phi)_{Q_k}] + \mu^2\|[\boldsymbol{\alpha}\cdot\boldsymbol{a}^\infty]\phi\|_{Q_k}^2 \end{aligned}$$

since $\|\boldsymbol{\alpha}\cdot D_0\phi\|_{Q_k}^2 = (\mathbb{P}_0(\mathbf{B}^o)\mathbb{I}_2\phi, \phi)_{Q_k}$. Therefore,

$$\begin{aligned} &\left|\left([(\mathbb{D}_{V_0} - V_0)^2 - 1 + W_0 - \mathbb{P}_{W_0}(\mathbf{B}^o)\mathbb{I}_2]\phi, \phi\right)_{Q_k}\right| \\ &\leq \theta(\mathbb{P}_0(\mathbf{B}^o)\mathbb{I}_2\phi, \phi) + (1+\tfrac{1}{\theta})\mu^2\|[\boldsymbol{\alpha}\cdot\boldsymbol{a}^\infty]\phi\|_{Q_k}^2 \end{aligned}$$

which completes the proof. □

It follows from Lemma 1 and (11) that for $\phi \in [C_0^\infty(Q_k)]^4$ and $\theta \in (0,1)$

$$\begin{aligned} &([(\mathbb{D}_{V_0}(\mathbf{B}) - V_0)^2 - 1 + W_0]\phi, \phi)_{Q_k} \leq \\ &\quad (1+\theta)(\mathbb{P}_0(\mathbf{B}^o)\mathbb{I}_2\phi, \phi)_{Q_k} + W_0\|\phi\|_{Q_k}^2 + \tfrac{C^2}{\theta}\mu^2 r^2\sigma_r^2\|\phi\|_{Q_k}^2 \end{aligned} \tag{16}$$

and

$$\begin{aligned} &([(\mathbb{D}_{V_0}(\mathbf{B}) - V_0)^2 - 1 + W_0]\phi, \phi)_{Q_k} \geq \\ &\quad (1-\theta)(\mathbb{P}_0(\mathbf{B}^o)\mathbb{I}_2\phi, \phi)_{Q_k} + W_0\|\phi\|_{Q_k}^2 - \tfrac{C^2}{\theta}\mu^2 r^2\sigma_r^2\|\phi\|_{Q_k}^2. \end{aligned} \tag{17}$$

Hence, from (13) and (14) we have that

$$\begin{aligned} &\tfrac{1}{2}\hbar^3 N([(\mathbb{D}_{V_0}(\mathbf{B}) - V_0)^2 - 1 + W_0]; Q_k) \geq \\ &\quad (1-\delta)^3\mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{W_0}{1+\theta} + \tfrac{C^2}{\theta(1+\theta)}\mu^2 r^2\sigma_r^2 + \tfrac{C\hbar^2}{\delta^2 r^2}; Q_k) \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\tfrac{1}{2}\hbar^3 N([(\mathbb{D}_{V_0}(\mathbf{B}) - V_0)^2 - 1 + W_0]; Q_k) \leq \\ &\quad \mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{W_0}{1-\theta} - \tfrac{C^2}{\theta(1-\theta)}\mu^2 r^2\sigma_r^2, Q_k). \end{aligned} \tag{19}$$

Similar estimates follow for the Pauli operator:

$$\begin{aligned} &\hbar^3 N(\mathbb{P}_{W_0}(\mathbf{B}); Q_k) \geq \\ &\quad (1-\delta)^3\mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{W_0}{1+\theta} + \tfrac{C^2}{\theta(1+\theta)}\mu^2 r^2\sigma_r^2 + \tfrac{C^2\hbar^2}{\delta^2 r^2}, Q_k) \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\hbar^3 N(\mathbb{P}_{W_0}(\mathbf{B}); Q_k) \leq \\ &\quad \mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{W_0}{1-\theta} - \tfrac{C^2}{\theta(1-\theta)}\mu^2 r^2\sigma_r^2, Q_k) \end{aligned} \tag{21}$$

.

To estimate error terms, we use the following inequalities for $\mathfrak{B} \equiv \mathfrak{B}_0$ established in Sobolev [23]: for any $\lambda \geq 0$ and any subset G of Ω

$$\begin{aligned} &|\mathfrak{B}(b, U_1+\lambda, G) - \mathfrak{B}(b, U_2+\lambda, G)| \\ &\lesssim \mathcal{M}(b, U_1-U_2, G) + \textstyle\int_G |U_1-U_2|^{1/2}(|U_1|+|U_2|)d\mathbf{x} \\ &\lesssim \mathcal{M}(b, U_1-U_2, G) + \mathcal{N}(U_1-U_2, G)^{\frac{1}{3}}\{\mathcal{N}(U_1-U_2, G)^{\frac{2}{3}} + \mathcal{N}(U_2, G)^{\frac{2}{3}}\}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} &|\mathfrak{B}(b_1, U, G) - \mathfrak{B}(b_2, U, G)| \lesssim \\ &\mathcal{M}(|b_1-b_2|, U, G) + \mathcal{M}(|b_1-b_2|, U, G)^{\frac{1}{2}}\mathcal{N}(U, G)^{\frac{1}{2}} \\ &+\mathcal{M}(|b_1-b_2|, U, G)^{\frac{1}{4}}\mathcal{N}(U, G)^{\frac{3}{4}} + \mathcal{M}(|b_1-b_2|, U, G)^{\frac{1}{2}}\mathcal{N}(U, G)^{\frac{1}{2}} \end{aligned} \tag{23}$$

where

$$\mathcal{M}(b, U, G) := \int_G b(x)|U(x)|^{\frac{1}{2}}dx, \ b \geq 0, \qquad \mathcal{N}(U, G) := \int_G |U(x)|^{\frac{3}{2}}dx.$$

Lemma 2. *If $r = A\hbar$, then*

(24)
$$\begin{aligned} &|Q_k|^{-1}\left|\mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{W_0}{1+\theta} + C_1\tfrac{\mu^2r^2\sigma_r^2}{\theta(1+\theta)} + C_2\tfrac{\hbar^2}{\delta^2r^2}, Q_k) - \mathfrak{B}(\mu\hbar|\mathbf{B}|, W_0, Q_k)\right| \\ &\lesssim \mu\hbar|\mathbf{B}^o|(\theta + \tfrac{A^2}{\theta}\mu^2\hbar^2\sigma_r^2 + \tfrac{1}{A^2\delta^2})^{\frac{1}{2}} + (\theta + \tfrac{A^2}{\theta}\mu^2\hbar^2\sigma_r^2 + \tfrac{1}{A^2\delta^2})^{\frac{3}{2}} + (\mu\hbar\sigma_r)^{\frac{1}{4}} + \mu\hbar\sigma_r. \end{aligned}$$

Proof. With $U_2 := W_0$ and

$$U_1 := \frac{W_0}{1+\theta} + C_1\frac{\mu^2r^2\sigma_r^2}{\theta(1+\theta)} + C_2\frac{\hbar^2}{\delta^2r^2},$$

$$\begin{aligned} |U_1 - U_2| \leq&\ \ |W_0|\theta + C_1\tfrac{\mu^2r^2\sigma_r^2}{\theta(1+\theta)} + C_2\tfrac{\hbar^2}{\delta^2r^2} \\ \lesssim&\ \ \theta + \tfrac{A^2}{\theta}\mu^2\hbar^2\sigma_r^2 + \tfrac{1}{A^2\delta^2} =: F(\theta, A, \mu\hbar, \delta, \sigma_r) \end{aligned}$$

and

$$|\mathbf{B} - \mathbf{B}^o| \leq \sigma_r, \qquad x \in Q_k.$$

Therefore,

$$\mathcal{M}(\mu\hbar|\mathbf{B}-\mathbf{B}^o|, W_0, Q_k) \lesssim \mu\hbar\sigma_r|Q_k|,$$

$$\mathcal{M}(\mu\hbar|\mathbf{B}^o|, U_1-U_2, Q_k) \lesssim \mu\hbar|\mathbf{B}^o|F(\theta, A, \mu, \hbar, \delta, \sigma_r)^{\frac{1}{2}}|Q_k|, \qquad \mathcal{N}(U_2, Q_k) \lesssim |Q_k|,$$

and

$$\mathcal{N}(|U_1-U_2|, Q_k) \lesssim F(\theta, A, \mu, \hbar, \delta, \sigma_r)^{\frac{3}{2}}|Q_k|.$$

The lemma follows from (20) and (21). □

3. The Pauli operator: $\mu\hbar \leq$ *constant*.

We assume that (A_1)-(A_3) hold in §2 with $V_0 \equiv 0$. It follows that $\mathbb{P}_{W_0}(\Omega) \leq \oplus_{k=1}^K \mathbb{P}_{W_0}(Q_k)$ in the form sense (see [[5],§XI.2.2]), which implies that

$$N(\mathbb{P}_{W_0}, \Omega) \geq \sum_{k=1}^K N(\mathbb{P}_{W_0}, Q_k), \tag{25}$$

and, since

$$
\begin{aligned}
M_\gamma(\mathbb{P}_{W_0}+\lambda,\Omega) &= -\int_0^\infty t^\gamma dN(\mathbb{P}_{W_0}+\lambda+t,\Omega) \\
&= \gamma\int_0^\infty t^{\gamma-1}N(\mathbb{P}_{W_0}+\lambda+t,\Omega)dt,
\end{aligned}
\tag{26}
$$

for all $\lambda > 0$, we have for all $\gamma \geq 0$

$$
M_\gamma(\mathbb{P}_{W_0}+\lambda,\Omega) \geq \sum_{k=1}^{K} M_\gamma(\mathbb{P}_{W_0}+\lambda,Q_k). \tag{27}
$$

Note that (26), and hence (27), hold for $\lambda = 0$ if $N(\mathbb{P}_{W_0},\Omega) < \infty$, which will be the case in Theorem 1 below, but not in Theorem 2 and Theorem 6. In Theorem 2 and Theorem 6 all we know is that $N(\mathbb{P}_W+\lambda,\Omega) = O(\lambda^{-1})$ and $O(\lambda^{-\frac{1}{2}})$ respectively, and as a consequence, we can claim that (26) holds for $\lambda = 0$ only if $\gamma > 1$ in the first case and $\gamma > \frac{1}{2}$ in the second. We are mainly interested in the remaining values of γ in Theorems 2 and 6, and take $\lambda > 0$.

Also, note that $\mathbb{P}_{W_0} \geq -(W_0)_- \geq -\Lambda$, say, and the inequalities (20) and (21) with W_0 replaced by $W_0+\lambda$ hold uniformly for $\lambda \in [0,\Lambda]$. Moreover,

$$
\mathfrak{B}_\gamma(b,W,G) = \gamma\int_0^\infty t^{\gamma-1}\mathfrak{B}_0(b,W+t,G)dt. \tag{28}
$$

Hence, from (18), (24), and (26), we see that when $\mu\hbar \lesssim 1$

$$
\begin{aligned}
&\hbar^3 M_\gamma(\mathbb{P}_{W_0}+\lambda;\Omega) \geq (1-\delta)^3\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,W_0+\lambda,Q) \\
&\quad -O\left(\left\{(\theta+\tfrac{A^2}{\theta}\sigma_r^2+\tfrac{1}{A^2\delta^2})^{\frac{1}{2}}+(\theta+\tfrac{A^2}{\theta}\sigma_r^2+\tfrac{1}{A^2\delta^2})^{\frac{3}{2}}+\sigma_r^{\frac{1}{4}}+\sigma_r\right\}|Q|\right).
\end{aligned}
$$

On allowing $\hbar \to 0$, $\theta \to 0$, $A \to \infty$, and $\delta \to 0$ in that order we have that for all $\gamma \geq 0$

$$
\liminf_{\hbar\to 0}\{\hbar^3 M_\gamma(\mathbb{P}_{W_0}+\lambda;\Omega) - \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,W_0+\lambda,\Omega)\} \geq 0. \tag{29}
$$

To prove the reverse inequality in (29), we proceed in the manner of Y. Colin de Verdiere [3] and Sobolev [23]. Let the interior of each Q_k be denoted by $int(Q_k)$. Set

$$
S := \mathbb{R}^3 \setminus \cup_{k=1}^K int(Q_k), \qquad S_{\rho r} := \{x\in\mathbb{R}^3 : dist(x,S) < \rho r\} \tag{30}
$$

for some $\rho \in (0,1)$. Construct a partition of unity $\{\psi_k\}_{k=0}^K$ subordinate to the covering $\cup_{k=1}^K int(Q_k)\cup S_{\rho r}$ of $\mathbb{R}^3$:

$$
\begin{array}{rl}
(i) & \psi_0 \in C^\infty(\mathbb{R}^3), \quad \psi_k \in C_0^\infty(int(Q_k)), k \geq 1, \\
(ii) & \sum_{k=0}^K \psi_k^2 \equiv 1, \\
(iii) & \sum_{k=0}^K |\nabla\psi_k(x)|^2 \lesssim (\rho r)^{-2}\chi_{\rho r},
\end{array}
\tag{31}
$$

where $\chi_{\rho r}$ is the characteristic function of $Q_{\rho r} := \{x\in\mathbb{R}^3 : dist(x,Q) < \rho r\}$. Note that

$$
|S_{\rho r}\cap Q| \lesssim \rho|Q|, \qquad |Q_{\rho r}\setminus Q| \lesssim \rho r. \tag{32}
$$

Then, for every $f \in [C_0^\infty(\Omega)]^2$, a calculation yields

$$
(\mathbb{P}_0\psi_0 f,\psi_0 f) + \sum_{k=1}^K (\mathbb{P}_0\psi_k f,\psi_k f) \leq (\mathbb{P}_0 f,f) + C\hbar^2(\rho r)^{-2}(\chi_{\rho r}f,f)
$$

and, as a consequence,

$$(\mathbb{P}_{W_0}(\Omega)f,f) \geq \begin{array}{l}([\mathbb{P}_{W_0}(S_{\rho r}\cap\Omega) - C\hbar^2(\rho r)^{-2}\chi_{\rho r}]\psi_0 f,\psi_0 f)\\ +\sum_{k=1}^K([\mathbb{P}_{W_0}(Q_k) - C\hbar^2(\rho r)^{-2}\chi_{\rho r}]\psi_k f,\psi_k f).\end{array} \tag{33}$$

By the minimax principle, $\lambda_n(\psi_k\mathbb{P}_{W_0}(Q)\psi_k) \geq \lambda_n(\mathbb{P}_{W_0}(Q))$ for $k=0,1,\ldots,K$, (see (21) of [10]). Therefore, it follows from (24) and (33) that for all $\gamma\geq 0$

$$M_\gamma(\mathbb{P}_{W_0}+\lambda,\Omega) \leq \begin{array}{l} M_\gamma(\mathbb{P}_{W_0}+\lambda - C\hbar^2(\rho r)^{-2}\chi_{\rho r}, S_{\rho r}\cap\Omega)\\ +\sum_{k=1}^K M_\gamma(\mathbb{P}_{W_0}+\lambda - C\hbar^2(\rho r)^{-2}\chi_{\rho r}, Q_k).\end{array} \tag{34}$$

From (21), we have in the case that $\mu\hbar \lesssim 1$, and uniformly for $t\in[0,\Lambda]$,

$$\begin{aligned}
&\hbar^3 N(\mathbb{P}_{W_0}+\lambda+t - C\hbar^2(\rho r)^{-2},Q_k)\\
&\leq \mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{W_0+\lambda+t}{1-\theta} - \tfrac{C^2}{\theta(1-\theta)}\mu^2r^2\sigma_r^2 - \tfrac{C\hbar^2}{(1-\theta)\rho^2r^2}, Q_k)\\
&\leq \mathfrak{B}(\mu\hbar|\mathbf{B}|, W_0+\lambda+t, Q_k) + O\Big(\big\{(\theta+\tfrac{A^2}{\theta}\sigma_r^2+\tfrac{1}{A^2\rho^2})^{\frac12}\\
&+(\theta+\tfrac{A^2}{\theta}\sigma_r^2+\tfrac{1}{A^2\rho^2})^{\frac32} + (\sigma_r)^{\frac14}+\sigma_r\big\}|Q_k|\Big)
\end{aligned}$$

as in Lemma 2. Hence, on using (26),

$$\begin{aligned}
&\hbar^3\textstyle\sum_{k=1}^K M_\gamma(\mathbb{P}_{W_0}+\lambda - C\hbar^2(\rho r)^{-2},Q_k)\\
&\leq \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|, W_0+\lambda, Q) + O\Big((\theta+\tfrac{A^2}{\theta}\sigma_r^2+\tfrac{1}{A^2\rho^2})^{\frac12}\\
&+(\theta+\tfrac{A^2}{\theta}\sigma_r^2+\tfrac{1}{A^2\rho^2})^{\frac32} + \sigma_r^{\frac14}+\sigma_r\Big).
\end{aligned} \tag{35}$$

For the first term on the right side of the inequality (34), we use the Cwikel-Lieb-Rozenblum inequality for a Schrödinger operator with a magnetic field[2] to derive the estimate

$$\begin{aligned}
&\hbar^3 M_\gamma(\mathbb{P}_{W_0}+\lambda - C\hbar^2(\rho r)^{-2}\chi_{\rho r}, S_{\rho r}\cap\Omega)\\
&\lesssim \textstyle\int_{S_{\rho r}\cap\Omega}[\mu\hbar|\mathbf{B}| - W_0 + \frac{\hbar^2}{(\rho r)^2}\chi_{\rho r}]_+^{\frac32+\gamma}d\mathbf{x}\\
&\lesssim \textstyle\int_{S_{\rho r}\cap\Omega}([|W_0|+|\mathbf{B}|]^{\frac32+\gamma} + [\frac{\hbar}{\rho r}]^{2\gamma+3}\chi_{\rho r})d\mathbf{x}\\
&\lesssim \textstyle\int_{\Omega\setminus Q}|\mathbf{B}|^{\frac32+\gamma}d\mathbf{x} + \int_{S_{\rho r}\cap Q}[|W_0|+|\mathbf{B}|]^{\frac32+\gamma}d\mathbf{x} + [\frac{\hbar}{(\rho r)}]^{2\gamma+3}|Q_{\rho r}\cap S_{\rho r}|\\
&\lesssim \textstyle\int_{\Omega\setminus Q}|\mathbf{B}|^{\frac32+\gamma}d\mathbf{x} + K(\rho r) + \frac{1}{(A\rho)^{2\gamma+3}}[\rho_+\rho r]
\end{aligned} \tag{36}$$

where

$$K(\rho r) = \int_{S_{\rho r}\cap Q}[|W_0|+|\mathbf{B}|]^{\frac32+\gamma}d\mathbf{x}\to 0, \qquad \text{as}\quad \rho\to 0,$$

uniformly in $\hbar$ and A, in view of (32). From (34), (35), and (36), on allowing $\hbar\to 0$, $\theta\to 0$, $A\to\infty$, and $\rho\to 0$, in that order, it follows that

$$\limsup_{\hbar\to 0}\big\{\hbar^3 M_\gamma(\mathbb{P}_{W_0}+\lambda,\Omega) - \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|, W_0+\lambda,\Omega)\big\} \lesssim \int_{\Omega\setminus Q}|\mathbf{B}(x)|^{\frac32+\gamma}dx. \tag{37}$$

Since $W_0=0$ outside Q, then Q can be chosen such that the right-hand side of (37) is arbitrarily small if $|\mathbf{B}|\in L^{3/2+\gamma}(\Omega)$. From this fact and (27), we conclude that for all $\gamma\geq 0$

$$\lim_{\hbar\to 0}\big\{\hbar^3 M_\gamma(\mathbb{P}_{W_0}+\lambda;\Omega) - \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|, W_0+\lambda,\Omega)\big\} = 0 \tag{38}$$

Now, we are in a position to prove the following general result for a Pauli operator with an electric potential that is not assumed to be piecewise constant.

[2]See Theorem 2.15 of Avron *et al.* [1] or Lemma 4.1 of [9].

Theorem 1. *Suppose that*
(i) $\mathbf{B}$ *is continuous on* Ω,
(ii) $W, |\mathbf{B}| \in L^{\frac{3}{2}+\gamma}(\Omega)$,
(iii) $\mu\hbar \ \le$ *constant.*

Then, for all $\gamma \ge 0$,

$$\lim_{\hbar\to 0}\{\hbar^3 M_\gamma(\mathbb{P}_W,\Omega) - \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|, W, \Omega)\} = 0$$

Proof. Given $\epsilon > 0$, there exists a finite collection of non-overlapping congruent cubes $\{Q_k\}_{k=1}^K$ for which (A_1)-(A_3) of §2 hold (for $V_0 \equiv 0$) with

$$\|W - W_0\|_{\frac{3}{2}+\gamma,\Omega} < \epsilon, \qquad \|\mathbf{B} - \mathbf{B}^o\|_{\frac{3}{2}+\gamma,\Omega} < \epsilon. \tag{39}$$

Note that W_0 depends on ε and hence so does the lower bound Λ of $\mathbb{P}_{W_0}$. Let $\eta \in (0,1)$ and set

$$\begin{aligned} \mathbb{P}_W =& \quad T_1 + T_2, \\ T_1 :=& \quad (1-\eta)\mathbb{P}_0 + W_0 - \eta\mu\hbar\boldsymbol{\sigma}\cdot\mathbf{B}^o \\ T_2 :=& \quad \eta\mathbb{P}_0 + W_\infty + \eta\mu\hbar\boldsymbol{\sigma}\cdot\mathbf{B}^o \end{aligned} \tag{40}$$

for $W_\infty := W - W_0$. Then,

$$\begin{aligned} T_1 =& \quad [\boldsymbol{\sigma}\cdot(-i\tilde{h}\nabla - \tilde{\mu}\boldsymbol{a})]^2 + W_0 - \eta\mu\hbar\boldsymbol{\sigma}\cdot\mathbf{B}^o \\ T_2 =& \quad \eta H_0(\mathbf{B}) + W_\infty - \eta\mu\hbar\boldsymbol{\sigma}\cdot\mathbf{B}^\infty \end{aligned}$$

where

$$\tilde{h} := (1-\eta)^{\frac{1}{2}}\hbar, \quad \tilde{\mu} := (1-\eta)^{\frac{1}{2}}\mu, \quad \text{and} \quad \mathbf{B}^\infty := \mathbf{B} - \mathbf{B}^o.$$

We have that

$$T_1 \ge S_1 := [\boldsymbol{\sigma}\cdot(-i\tilde{h}\nabla - \tilde{\mu}\boldsymbol{a})]^2 + W_0 - \eta\mu\hbar|\mathbf{B}^o|$$

and

$$T_2 \ge S_2 := \eta H_0(\mathbf{B}) + W_\infty - \eta\mu\hbar|\mathbf{B}^\infty|.$$

It then follows that

$$M_\gamma(\mathbb{P}_W,\Omega) \le M_\gamma(S_1,\Omega) + M_\gamma(S_2,\Omega). \tag{41}$$

From (38) with $\lambda = 0$ (which, according to the remark after (27), is allowed since $N(S_1,\Omega) < \infty$)

$$\lim_{\hbar\to 0}\{\tilde{h}^3 M_\gamma(S_1,\Omega) - \mathfrak{B}_\gamma(\tilde{\mu}\tilde{h}|\mathbf{B}|, W_0 - \eta\mu\hbar|\mathbf{B}^o|,\Omega)\} = 0. \tag{42}$$

and by Theorem 2.15 of Avron *et al.* [1]

$$\hbar^3 M_\gamma(S_2,\Omega) \lesssim \int_\Omega [\eta^{-1}W_\infty - \mu\hbar|\mathbf{B}^\infty|]_-^{\frac{3}{2}+\gamma}dx \lesssim (\epsilon/\eta)^{\frac{3}{2}+\gamma}. \tag{43}$$

Let $\gamma = 0$. On using (22), we have that

$$\begin{aligned} &|\mathfrak{B}(\tilde{\mu}\tilde{h}|\mathbf{B}|, W_0 - \eta\mu\hbar|\mathbf{B}^o|,\Omega) - \mathfrak{B}(\mu\hbar|\mathbf{B}|, W, \Omega)| \\ &\lesssim \int_\Omega |\mathbf{B}|[|W_\infty| + \eta|\mathbf{B}^o|]^{\frac{1}{2}}dx \\ &\quad + \|W_\infty + \eta|\mathbf{B}^o|\|_{\frac{3}{2},\Omega}^{\frac{1}{2}}\{\|W_\infty + \eta|\mathbf{B}^o|\|_{\frac{3}{2},\Omega} + \|W\|_{\frac{3}{2},\Omega}\}. \end{aligned} \tag{44}$$

It follows from (40)-(44) that

$$\limsup_{\hbar\to 0}\{\hbar^3 N(\mathbb{P}_W;\Omega) - \mathfrak{B}(\mu\hbar|\mathbf{B}|, W, \Omega)\} \le 0.$$

The cases $\gamma > 0$ follow similarly from (26) and the inequalities (2.23) - (2.25) for $\mathbf{B}_\gamma$ in [23]. The reverse inequality is obtained by choosing $\eta \in (-1, 0)$ and repeating the argument. □

If $|\mathbf{B}|$ is not in $L^{3/2}(\Omega)$, there may be an infinite number of negative eigenvalues. In this case we have the following

Theorem 2. *Suppose that*

1. $\mathbf{B}$ *is continuous,*
2. $|\mathbf{B}|$, $W \in L^\infty(\Omega)$,
3. *for* $p > 3/2$, $W^{5/2}$, $b_p^{3/2} W \in L^1(\Omega)$, *where* b_p *is defined in (9),*
4. $\mu\hbar \leq$ *constant.*

Then, for all $\gamma \in [0, 1)$ *and* $\lambda > 0$,

$$\lim_{\hbar \to 0} \left\{ \hbar^3 M_\gamma(\mathbb{P}_W + \lambda; \Omega) - \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|, W + \lambda, \Omega) \right\} = 0. \tag{45}$$

If $\gamma \geq 1$, *(45) with* $\lambda = 0$ *is proved in* [23].

Proof. We first note that $\mathbb{P}_W$ is properly defined as a form sum by Proposition 3. Also, from [[20], Theorem 1.1] for $\lambda > 0$,

$$\begin{aligned} N(\mathbb{P}_W + \lambda, \mathbb{R}^3) &\leq |\lambda|^{-1} M_1(\mathbb{P}_W, \mathbb{R}^3) \\ &\lesssim |\lambda|^{-1} \{\hbar^{-3} \int_{\mathbb{R}^3} W_-^{5/2} d\mathbf{x} + \mu^{3/2} \hbar^{-3/2} \int_{\mathbb{R}^3} b_p^{3/2} W_- d\mathbf{x}\} \end{aligned} \tag{46}$$

and hence, for any $\gamma \geq 0$ and $\lambda > 0$

$$M_\gamma(\mathbb{P}_W + \lambda, \Omega) \lesssim |\lambda|^{-1} \{\hbar^{-3} \int_\Omega W_-^{5/2} d\mathbf{x} + \mu^{3/2} \hbar^{-3/2} \int_\Omega b_p^{3/2} W_- d\mathbf{x}\} \tag{47}$$

Suppose that W_0, $\mathbf{B}^0$, $Q = \cup_{k=1}^K Q_k$ satisfy (A_1)-(A_3) of §2. Then (29) follows as before, and so do (34) and (35), with W_0 replaced by $W_0 + \lambda$. The remaining term on the right-hand side of (34) is estimated by (46). Since $(W_0 + \lambda)_- \leq (W_0)_-$, we have, with $r = A\hbar$,

$$\begin{aligned} &|\lambda| \hbar^3 M_\gamma(\mathbb{P}_{W_0} + \lambda - C\hbar^2 (\rho r)^{-2} \chi_{\rho r}, S_{\rho r} \cap \Omega) \\ &\lesssim \textstyle\int_{S_{\rho r} \cap \Omega} \left([W_0 - C\hbar^2 (\rho r)^{-2} \chi_{\rho r}]_-^{5/2} + b_p^{3/2} [W_0 - C\hbar^2 (\rho r)^{-2} \chi_{\rho r}]_- \right) d\mathbf{x} \\ &\lesssim \textstyle\int_{S_{\rho r} \cap \Omega} \left([W_0]_-^{5/2} + b_p^{3/2} [W_0]_- \right) d\mathbf{x} \\ &\quad + (\hbar/\rho r)^5 |S_{\rho r} \cap Q_{\rho r}| + (\hbar/\rho r)^2 \textstyle\int_{S_{\rho r} \cap Q_{\rho r}} b_p^{3/2} d\mathbf{x} \\ &\lesssim K(\rho r) + (1/A\rho)^5 (\rho + \rho r) + (1/A\rho)^2 (\rho + \rho r), \end{aligned}$$

where $K(\rho r) \to 0$ as $\rho \to 0$, uniformly in $\hbar$ and A, by (32). It follows as for (38) that

$$\lim_{\hbar \to 0} \left\{ \hbar^3 M_\gamma(\mathbb{P}_{W_0} + \lambda; \Omega) - \mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|, W_0 + \lambda, \Omega) \right\} = 0. \tag{48}$$

For a general W we proceed in a similar way to Sobolev in [23], using Shen's estimate (46).

Let $\varepsilon > 0$ and choose non-overlapping cubes Q_k, $k = 1, ..., K$, with sides parallel to the co-ordinate planes, and a piecewise constant W_0 which is constant in each Q_k, is zero outside $Q = \cup_{k=1}^K Q_k$ and

$$\int_\Omega |W - W_0|^{3/2} d\mathbf{x} < \varepsilon, \int_\Omega b_p |W - W_0|^{1/2} d\mathbf{x} < \varepsilon. \tag{49}$$

Note that assumptions 2. and 3. imply that $W^{3/2}$, $b_p W^{1/2} \in L^1(\Omega)$ and

$$\int_\Omega |W - W_0|^{5/2} d\mathbf{x} \lesssim \varepsilon, \quad \int_\Omega b_p^{3/2} |W - W_0| d\mathbf{x} \lesssim \varepsilon. \tag{50}$$

We prove the result for $\gamma = 0$, the proof for $\gamma > 0$ being similar. Set $N_\lambda(W) \equiv N(\mathbb{P}_W + \lambda, \Omega)$. Then, we have the Weyl inequality

$$N_\lambda(W) \geq N_\lambda(W_0/(1+\zeta)) - N_\lambda([W - W_0]/\zeta)$$

for any $\zeta \in (0,1)$. By (46) and (50)

$$\hbar^3 N_\lambda([W - W_0]/\zeta) \lesssim |\lambda|^{-1}\big(\varepsilon\zeta^{-5/2} + \varepsilon\zeta^{-1}\big). \tag{51}$$

Also, from (48),

$$\lim_{\hbar \to 0} \big\{\hbar^3 N_\lambda(W_0/(1+\zeta)) - \mathfrak{B}_0(\mu\hbar|\mathbf{B}|, W_0/(1+\zeta) + \lambda, \Omega)\big\} = 0$$

and from (22) we have

$$|\mathfrak{B}_0(\mu\hbar|\mathbf{B}|, W_0/(1+\zeta) + \lambda, \Omega) - \mathfrak{B}_0(\mu\hbar|\mathbf{B}|, W + \lambda, \Omega)| \lesssim \zeta^{1/2} + \varepsilon^{1/2}.$$

Hence,

$$\begin{gathered}\liminf_{\hbar \to 0} \big\{\hbar^3 N_\lambda(W) - \mathfrak{B}_0(\mu\hbar|\mathbf{B}|, W + \lambda, \Omega)\big\} \gtrsim \\ -|\lambda|^{-1}(\varepsilon\zeta^{-5/2} + \varepsilon\zeta^{-1}) - \zeta^{1/2} - \varepsilon^{1/2}.\end{gathered}$$

Let $\varepsilon \to 0$, $\zeta \to 0$ in that order to get

$$\liminf_{\hbar \to 0} \big\{\hbar^3 N_\lambda(W) - \mathfrak{B}_0(\mu\hbar|\mathbf{B}|, W + \lambda, \Omega)\big\} \geq 0.$$

The proof of the upper bound is similar. We use

$$N_\lambda(W) \leq N_\lambda(W_0/(1-\zeta)) + N_\lambda([W - W_0]/\zeta)$$

and argue as before. □

The following result can be established similarly.

Theorem 3. *Suppose that*

(i) W, $\mathbf{B}$ *are continuous,*
(ii) $|W(\mathbf{x})| \to 0$ *uniformly as* $|\mathbf{x}| \to \infty$ *in* Ω, *and*
(iii) $\mu\hbar \leq$ *constant.*

Then, (45) holds for all $\gamma \in [0,1)$ *and* $\lambda > 0$.

Proof. The operator $\mathbb{P}_W(\Omega)$ is properly defined as a self-adjoint operator on the domain of $\mathbb{P}_0(\Omega)$ since it is a bounded perturbation of $\mathbb{P}_0(\Omega)$. The point to note is that $[W + \lambda]_-$ is compactly supported in Ω. On choosing Q in (A_1) of §2 to contain this support, and the piecewise constant function W_0 to be such that $W_\infty \equiv W - W_0 \geq 0$ in Q, we have for

$$\mathbb{P}_W + 2\lambda = \{(1-\eta)\mathbb{P}_0 + W_0 + \lambda\} + \{\eta\mathbb{P}_0 + W_\infty + \lambda\} =: \mathbb{R}_1 + \mathbb{R}_2$$

that $\mathbb{R}_2 \geq 0$ for all $\eta \in (0,1)$. The proof follows easily from (48). □

4. The Dirac operator: $\mu\hbar \le$ *constant*

Assume (A_1)-(A_3) of §2 with $W_0 \equiv 0$. Set $N(S,\Omega,I) := \#\{\lambda_n(S) : \lambda_n(S) \in I\}$, the number of eigenvalues of the operator $S(\Omega)$ in the interval I, and $M_\gamma(S;\Omega,I) = \sum_{\lambda_n(S)\in I} |\lambda_n(S)|^\gamma$. Then, with $V_0 \ge 0$ in Q_k,

$$\begin{aligned}
&N(\mathbb{D}_{V_0}, Q_k, (-1,1)) + N(\mathbb{D}_{-V_0}, Q_k, (-1,1))\\
&= N(\mathbb{D}_{V_0}, Q_k, (-1,1)) + N(\mathbb{D}_{V_0}, Q_k, (-1+2V_0, 1+2V_0))\\
&= N(\mathbb{D}_{V_0}, Q_k, (-1, 1+2V_0)) + E,
\end{aligned}$$

where

$$E := \begin{cases} N(\mathbb{D}_{V_0}, Q_k, (-1+2V_0, 1)) = 0 & \text{if } \ V_0 \le 1,\\ -N(\mathbb{D}_{V_0}, Q_k, [1, -1+2V_0)) & \text{if } \ V_0 > 1. \end{cases}$$

For $\mathbb{R}_0 := \mathbb{D}_0^2 - 1 = \mathbb{P}_0 \mathbb{I}_2$ and $\lambda_1^o := (1+V_0)^2 - 1$,

$$N(\mathbb{D}_{V_0}, Q_k, (-1, 1+2V_0)) = N(\mathbb{R}_0, Q_k, [-1, \lambda_1^o)),$$

which implies that for any $\eta > 0$

$$\begin{aligned}
&|N(\mathbb{D}_{V_0}, Q_k, (-1,1)) + N(\mathbb{D}_{-V_0}, Q_k, (-1,1)) - N(\mathbb{R}_0, Q_k, [-1, \lambda_1^o))|\\
&\le N(\mathbb{R}_0, Q_k, [-1, \lambda_{-1}^o + \eta))
\end{aligned} \tag{52}$$

where $\lambda_{-1}^o := (1-V_0)^2 - 1$. Inequality (52) holds as well for $V_0 < 0$ with

$$\lambda_1^o := (1+|V_0|)^2 - 1, \qquad \lambda_{-1}^o := (1-|V_0|)^2 - 1. \tag{53}$$

For $\mathbb{R}_{V_0} \equiv \mathbb{R}_{V_0}(\mathbf{B}) := {\mathbb{D}_{V_0}}^2 - 1$, we have $\mathbb{R}_{V_0}(\Omega) \le \oplus_{k=1}^K \mathbb{R}_{V_0}(Q_k)$. Since

$$N(\mathbb{D}_{V_0}, \Omega, I) = N(\mathbb{R}_{V_0}, \Omega) = N(\mathbb{R}_{V_0}, \Omega, (-\infty, 0)), \qquad I = (-1,1), \tag{54}$$

we may apply the minimax principle to $\mathbb{R}_{V_0}$ in order to estimate $N(\mathbb{D}_{V_0}, \Omega, I)$. In particular,

$$\begin{aligned}
N(\mathbb{D}_{V_0}, \Omega, I) + N(\mathbb{D}_{-V_0}, \Omega, I) \ge\ & \textstyle\sum_{k=1}^K \{N(\mathbb{R}_{V_0}, Q_k) + N(\mathbb{R}_{-V_0}, Q_k)\}\\
=\ & \textstyle\sum_{k=1}^K \{N(\mathbb{D}_{V_0}, Q_k, I) + N(\mathbb{D}_{-V_0}, Q_k, I)\}.
\end{aligned}$$

Hence, from (18), (19), and (52)

$$\begin{aligned}
&\tfrac{1}{2}\hbar^3 \{N(\mathbb{D}_{V_0}, \Omega, I) + N(\mathbb{D}_{-V_0}, \Omega, I)\}\\
&\ge \tfrac{1}{2}\hbar^3 \textstyle\sum_{k=1}^K \{N(\mathbb{R}_0, Q_k, [-1, \lambda_1^o)) - N(\mathbb{R}_0, Q_k, [-1, \lambda_{-1}^o + \eta))\}\\
&\ge (1-\delta)^3 \mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{-\lambda_1^o}{1+\theta} + \tfrac{(C\mu r\sigma_r)^2}{\theta(1+\theta)} + \tfrac{C\hbar^2}{\delta^2 r^2}, Q)\\
&\quad -\mathfrak{B}(\mu\hbar|\mathbf{B}^o|, \tfrac{-\lambda_{-1}^o - \eta}{1-\theta} - \tfrac{(C\mu r\sigma_r)^2}{\theta(1-\theta)}, Q)\\
&= (1-\delta)^3 \mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_1^o, Q) - \mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_{-1}^o, Q)\\
&\quad -O\Big(\big\{(\theta + \eta + \tfrac{A^2\sigma_r^2}{\theta} + \tfrac{1}{A^2\delta^2})^{\frac{1}{2}} + (\theta + \eta + \tfrac{A^2\sigma_r^2}{\theta} + \tfrac{1}{A^2\delta^2})^{\frac{3}{2}} + \sigma_r^{\frac{1}{4}} + \sigma_r\big\}|Q|\Big)
\end{aligned}$$

for $r = A\hbar$, on using Lemma 14 with $\mu\hbar \lesssim 1$. It follows as for (29) that

$$\begin{aligned}
\liminf_{\hbar\to 0}\ &\big\{\tfrac{1}{2}\hbar^3 [N(\mathbb{D}_{V_0}, \Omega, I) + N(\mathbb{D}_{-V_0}, \Omega, I)]\\
&-\mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_1^o, \Omega) + \mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_{-1}^o, \Omega)\big\} \ge 0.
\end{aligned} \tag{55}$$

Note that $\lambda_{-1}^o \le 0$ if $|V_0| \le 2$, and consequently,

$$\mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_{-1}^o, Q_k) = 0 \quad \text{if} \ \ |V_0| \le 2. \tag{56}$$

To establish the reverse inequality in (55), we first note that for $\{\psi_k\}_{k=0}^K$ given in (31) and all $f \in [C_0^\infty(\Omega)]^4$

$$\mathbb{D}_{V_0}{}^2[\psi_k f] = \psi_k \mathbb{D}_{V_0}{}^2 f + \frac{\hbar}{i}\sum_{j=1}^3 \partial_j\psi_k[\alpha_j\mathbb{D}_{V_0}f + \mathbb{D}_{V_0}(\alpha_j f)] - \hbar^2(\Delta\psi_k)f.$$

Since $\sum_{k=0}^K \psi_k^2 \equiv 1$, it follows that

$$\begin{aligned}\textstyle\sum_{k=0}^K \psi_k\mathbb{D}_{V_0}{}^2[\psi_k f] = &\quad \mathbb{D}_{V_0}{}^2 f - \hbar^2 \textstyle\sum_{k=0}^K \psi_k(\Delta\psi_k)f\\ = &\quad \mathbb{D}_{V_0}{}^2 f + \hbar^2\textstyle\sum_{k=0}^K|\nabla\psi_k|^2 f.\end{aligned}$$

Therefore, by (31)

$$\begin{aligned}(\mathbb{R}_{V_0}(\Omega)f,f) \geq &\quad \textstyle\sum_{k=1}^K([\mathbb{R}_{V_0}(Q_k) - \frac{C\hbar^2}{\rho^2 r^2}]\psi_k f, \psi_k f)\\ &+([\mathbb{R}_{V_0}(S_{\rho r}\cap\Omega) - \frac{C\hbar^2}{\rho^2r^2}\chi_{\rho r}]\psi_0 f,\psi_0 f).\end{aligned}\tag{57}$$

Let $a^2 := C\hbar^2/(\rho r)^2$, where C is the constant in (57). Then,

$$N(\mathbb{R}_{V_0} - a^2, Q_k) = N(\mathbb{D}_{V_0}, Q_k, (-\sqrt{1+a^2}, \sqrt{1+a^2})),$$

and on repeating the argument leading to (52), we have for any $\eta > 0$, with $\lambda_{\pm 1}^o(a) = (\sqrt{1+a^2} \pm |V_0|)^2 - 1$,

(58)

$$\begin{aligned}&\tfrac{1}{2}\hbar^3\{N(\mathbb{R}_{V_0} - a^2, Q_k) + N(\mathbb{R}_{-V_0} - a^2, Q_k)\}\\ &\leq \tfrac{1}{2}\hbar^3\{N(\mathbb{R}_0 - \lambda_1^o(a), Q_k) + N(\mathbb{R}_0 - \lambda_{-1}^o(a) - \eta, Q_k)\}\\ &\leq \mathfrak{B}(\mu\hbar|\mathbf{B}^o|, -\tfrac{\lambda_1^o(a)}{1-\theta} - \tfrac{(C\mu r\sigma_r)^2}{\theta(1-\theta)}, Q_k) + \mathfrak{B}(\mu\hbar|\mathbf{B}^o|, -\tfrac{\lambda_{-1}^o(a)}{1-\theta} - \tfrac{\eta}{1-\theta} - \tfrac{(C\mu r\sigma_r)^2}{\theta(1-\theta)}, Q_k)\\ &\leq \mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_1^o, Q_k) + \mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_{-1}^o, Q_k)\\ &+O\Big(\big\{(\theta+\eta+\tfrac{1}{A\rho}+\tfrac{A^2\sigma_r^2}{\theta})^{\frac12}\\ &\qquad +(\theta+\eta+\tfrac{1}{A\rho}+\tfrac{A^2\sigma_r^2}{\theta}+)^{\frac32}+\sigma_r^{\frac14}+\sigma_r\big\}|Q_k|\Big)\end{aligned}$$

by (19) and (24), since $r = A\hbar$. From Lemma 4.1 of [9]

(59)

$$\begin{aligned}\hbar^3 N(\mathbb{R}_{V_0} - \tfrac{C\hbar^2}{\rho^2r^2}\chi_{\rho r}, S_{\rho r}\cap\Omega) \lesssim &\quad \textstyle\int_{S_{\rho r}\cap\Omega}[|V_0|^{\frac32} + |V_0|^3 + |\mathbf{B}|^{\frac32} + \frac{\hbar^3}{(\rho r)^3}\chi_{\rho r}]d\mathbf{x}\\ \lesssim &\quad \textstyle\int_{\Omega\setminus Q}|\mathbf{B}(x)|^{\frac32}d\mathbf{x} + K(\rho r) + (A\rho)^{-3}[\rho + \rho r]\end{aligned}$$

where

$$K(\rho r) := \int_{S_{\rho r}\cap Q}[|V_0|^{\frac32} + |V_0|^3|\mathbf{B}|^{\frac32}]d\mathbf{x} \to 0, \qquad \text{as} \quad \rho \to 0,$$

uniformly in $\hbar$ and A by (32). As in (37), on using (54), it follows from (57), (58), and (59) that

$$\begin{aligned}\limsup_{\hbar\to 0} &\quad \big\{\tfrac12\hbar^3[N(\mathbb{D}_{V_0},\Omega,I) + N(\mathbb{D}_{-V_0},\Omega,I)]\\ &\quad -\mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_1^o, \Omega) - \mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_{-1}^o, \Omega)\big\} \leq 0\end{aligned}\tag{60}$$

for $I = (-1, 1)$.

Now, we are in a position to prove the first theorem in this section.

Theorem 4. *Suppose that*

(i) $\mathbf{B}$ *is continuous on* Ω,
(ii) $V, V^2, |\mathbf{B}| \in L^{\frac32+\gamma}(\Omega)$,
(iii) $|\{\mathbf{x}\in\Omega : |V(\mathbf{x})| > 2\}| = 0$,
(iv) $\mu\hbar \leq$ *constant.*

Then, with $I_\eta := (-1+\eta, 1-\eta)$, $\eta \in (0,1)$, *and any* $\gamma \in [0,1]$,

$$\lim_{\eta\to 0}\lim_{\hbar\to 0}\{\frac{1}{2}\hbar^3[M_\gamma(\mathbb{D}_V;\Omega,I_\eta)+M_\gamma(\mathbb{D}_{-V};\Omega,I_\eta)]-\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,-\lambda_1,\Omega)\}=0$$

where $\lambda_1 := 2|V| + V^2$.

Proof. We prove the result for $\gamma = 0$, the other cases being similar. As in the proof of Theorem 1, given $\epsilon > 0$ choose a piecewise constant function V_0 taking constant values in non-overlapping congruent cubes Q_k, $k = 1, \dots, K$, and such that

$$\|V-V_0\|_{\frac{3}{2},\Omega}<\epsilon, \quad \|V-V_0\|_{3,\Omega}<\epsilon \quad \|\mathbf{B}-\mathbf{B}^o\|_{\frac{3}{2},\Omega}<\epsilon \tag{61}$$

in which $V_0 \equiv 0$, $\mathbf{B}^o \equiv 0$ outside $Q := \cup_{k=1}^K Q_k$. Thus, (52) and (60) hold for V_0, and, indeed, for any function taking constant values in each Q_k and zero outside Q. Let $\eta \in (0,1)$ and set

$$\begin{aligned}\mathbb{D}_V = & \quad (1-\eta)\{\boldsymbol{\alpha}\cdot(\tfrac{\hbar}{i}\nabla-\mu\boldsymbol{a})\}+\beta+V+\eta\{\boldsymbol{\alpha}\cdot(\tfrac{\hbar}{i}\nabla-\mu\boldsymbol{a})\}\\ = & \quad \{\boldsymbol{\alpha}\cdot(\tfrac{\hat{h}}{i}\nabla-\hat{\mu}\boldsymbol{a})\}+\beta+V_0\}+\{\eta\boldsymbol{\alpha}\cdot(\tfrac{\hbar}{i}\nabla-\mu\boldsymbol{a})+V_\infty\}\\ =: & \quad T_1+T_2\end{aligned}\tag{62}$$

where $\hat{h} := (1-\eta)\hbar$, $\hat{\mu} := (1-\eta)\mu$, and $V_\infty = V - V_0$. Then, for $\phi \in [C_0^\infty(\Omega)]^4$,

$$([\mathbb{D}_V^2-1]\phi,\phi)=\|T_1\phi\|^2+2\mathfrak{Re}(T_1\phi_1,T_2\phi)+\|T_2\phi\|^2-\|\phi\|^2. \tag{63}$$

With $\hat{D}_{\boldsymbol{a}} := \frac{\hat{h}}{i}\nabla - \hat{\mu}\boldsymbol{a}$ and $D_{\boldsymbol{a}} := \frac{\hbar}{i}\nabla - \mu\boldsymbol{a}$, we have

$$\begin{aligned}(T_1\phi,T_2\phi)= & \quad (\{\boldsymbol{\alpha}\cdot\hat{D}_{\boldsymbol{a}}+\beta+V_0\}\phi,\{\tfrac{\eta}{1-\eta}\boldsymbol{\alpha}\cdot\hat{D}_{\boldsymbol{a}}+V_\infty\}\phi)\\ = & \quad \tfrac{\eta}{1-\eta}\|T_1\phi\|^2+(T_1\phi,[V_\infty-\tfrac{\eta}{1-\eta}(\beta+V_0)]\phi).\end{aligned}$$

Since $\mathfrak{Re}(\boldsymbol{\alpha}\cdot\hat{D}_{\boldsymbol{a}}\phi,\beta\phi)=0$,

$$\begin{aligned}&2\mathfrak{Re}(T_1\phi,T_2\phi)=\\ &\quad\tfrac{2\eta}{1-\eta}\|T_1\phi\|^2+2\mathfrak{Re}[(T_1\phi,V_\infty\phi)-\tfrac{2\eta}{1-\eta}(T_1\phi,V_0\phi)-\tfrac{2\eta}{1-\eta}((\beta+V_0)\phi,\beta\phi)].\end{aligned}$$

Hence,

$$\begin{aligned}&|2\mathfrak{Re}(T_1\phi,T_2\phi)-\tfrac{2\eta}{1-\eta}\|T_1\phi\|^2+\tfrac{2\eta}{1-\eta}\|\phi\|^2|\\ &\le 2\theta\|T_1\phi\|^2+\tfrac{1}{\theta}(\|V_\infty\phi\|^2+\tfrac{\eta^2}{(1-\eta)^2}\|V_0\phi\|^2)+\tfrac{2\eta}{1-\eta}(|V_0|\phi,\phi).\end{aligned}\tag{64}$$

Also,

$$\|T_2\phi\|^2=\eta^2\|\boldsymbol{\alpha}\cdot D_{\boldsymbol{a}}\phi\|^2+2\eta\mathfrak{Re}(\boldsymbol{\alpha}\cdot D_{\boldsymbol{a}}\phi,V_\infty\phi)+\|V_\infty\phi\|^2$$

which implies that for any $\theta > 0$

$$\begin{aligned}&|\|T_2\phi\|^2-\eta^2([H_0(\mathbf{B})-\mu\hbar\sigma\cdot\mathbf{B}]\mathbb{I}_2\phi,\phi)|\\ &\le\theta\eta^2([H_0(\mathbf{B})-\mu\hbar\sigma\cdot\mathbf{B}]\mathbb{I}_2\phi,\phi)+(1+\tfrac{1}{\theta})|V_\infty\phi\|^2.\end{aligned}\tag{65}$$

On substituting (64) and (65) in (63), we have that

$$\begin{aligned}([\mathbb{D}_V^2-1]\phi,\phi)\ge & \quad (1+\tfrac{2\eta}{1-\eta}-2\theta)\|T_1\phi\|^2-\tfrac{1}{\theta}(\|V_\infty\phi\|^2+\tfrac{\eta^2}{(1-\eta)^2}\|V_0\phi\|^2)\\ & \quad -\tfrac{2\eta}{1-\eta}(|V_0|\phi,\phi)+\eta^2(1-\theta)([H_0(\mathbf{B})-\mu\hbar\sigma\cdot\mathbf{B}]\mathbb{I}_2\phi,\phi)\\ & \quad -(1+\tfrac{1}{\theta})\|V_\infty\phi\|^2-\|\phi\|^2-\tfrac{2\eta}{1-\eta}\|\phi\|^2\\ \ge & \quad (1+\tfrac{2\eta}{1-\eta}-2\theta)([\hat{\mathbb{D}}_{V_0}^2-1]\phi,\phi)-\tfrac{\eta^2}{\theta(1-\eta)^2}\|V_0\phi\|^2\\ & \quad -\tfrac{2\eta}{1-\eta}(|V_0|\phi,\phi)-\eta^2(1-\theta)\mu\hbar(|\mathbf{B}^o|\phi,\phi)\\ & \quad +\eta^2(1-\theta)([H_0(\mathbf{B})-\mu\hbar|\mathbf{B}^\infty|]\mathbb{I}_2\phi,\phi)\\ & \quad -(1+\tfrac{2}{\theta})(V_\infty^2\phi,\phi)-2\theta\|\phi\|^2\end{aligned}\tag{66}$$

where $\hat{\mathbb{D}}_{V_0} = T_1 = \boldsymbol{\alpha}\cdot(\frac{\hat{h}}{i}\nabla - \hat{\mu}\boldsymbol{a}) + \beta + V_0$. Whence, on choosing $\theta = \eta/(1-\eta) < 1/2$,

$$\begin{aligned} \mathbb{D}_V^2 - (1 - \tfrac{2\eta}{1-\eta}) \geq \quad & \hat{\mathbb{D}}_{V_0}^2 - 1 - \tfrac{\eta}{1-\eta}(V_0^2 + 2|V_0|) - \eta^2\mu\hbar|\mathbf{B}^o| \\ & + \tfrac{\eta^2}{2}[H_0(\mathbf{B}) - \mu\hbar|\mathbf{B}^\infty|]\mathbb{I}_2 - \tfrac{2-\eta}{\eta}V_\infty^2 \\ =: \quad & S_1 + S_2 \end{aligned} \tag{67}$$

with

$$S_1 \equiv S_1(V_0) := \hat{\mathbb{D}}_{V_0}^2 - 1 - \Phi, \qquad \Phi := \frac{\eta}{1-\eta}(V_0^2 + 2|V_0|) + \eta^2\mu\hbar|\mathbf{B}^o|,$$

and

$$S_2 := \frac{\eta^2}{2}[H_0(\mathbf{B}) - \mu\hbar|\mathbf{B}^\infty| - \frac{2(2-\eta)}{\eta^3}V_\infty^2]\mathbb{I}_2.$$

Since Φ is piecewise constant and vanishes outside Q, the analysis leading to inequality (58) holds with $\mathbb{R}_{V_0} = \hat{\mathbb{D}}_{V_0}^2 - 1$ and $a^2 = \Phi$, and this yields the estimate

$$\begin{aligned} & \tfrac{1}{2}\hat{h}^3\{N(S_1(V_0),\Omega) + N(S_1(-V_0),\Omega)\} \\ & \leq \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_1^o, \Omega) + \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_{-1}^o, \Omega) \\ & + O\big([\theta + \eta^{\frac{1}{2}} + \tfrac{A^2\sigma_r^2}{\theta}]^{\frac{1}{2}} + [\theta + \eta^{\frac{1}{2}} + \tfrac{A^2\sigma_r^2}{\theta}]^{\frac{3}{2}} + \sigma_r^{\frac{1}{4}} + \sigma_r\big) \end{aligned}$$

where $\lambda_{\pm 1}^o = (1 \pm |V_0|)^2 - 1$, θ is arbitrary and $r = Ah$. Also, by (22)

$$|\mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_{\pm 1}^o, \Omega) - \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_{\pm 1}, \Omega)| = O(\|V_\infty\|_{\frac{3}{2},\Omega}^{\frac{1}{2}} + \|V_\infty\|_{3,\Omega} + \|\mathbf{B}\|_{\frac{3}{2},\Omega}) \lesssim \varepsilon^{1/2}$$

with $\lambda_{\pm 1} := (1 \pm |V|)^2 - 1$. It follows that

$$\begin{aligned} & \limsup_{\hbar\to 0}\big\{\tfrac{1}{2}\hat{h}^3[N(S_1(V_0),\Omega) + N(S_1(-V_0),\Omega)] \\ & \quad - \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_1, \Omega) - \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_{-1}, \Omega)\big\} \lesssim \epsilon^{\frac{1}{2}} + \eta^{\frac{1}{4}}. \end{aligned} \tag{68}$$

A similar argument yields the reverse inequality (c.f.(55))

$$\begin{aligned} & \liminf_{\hbar\to 0}\big\{\tfrac{1}{2}\hat{h}^3\{N(S_1(V_0),\Omega) + N(S_1(-V_0),\Omega)\} \\ & \quad - \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_1, \Omega) + \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_{-1}, \Omega)\big\} \gtrsim -(\epsilon^{\frac{1}{2}} + \eta^{\frac{1}{4}}). \end{aligned} \tag{69}$$

Note that $\lambda_{-1} = (1 - |V|)^2 - 1 \leq 0$ for $|V| \leq 2$, and also $[2k\hat{\mu}\hat{h}|\mathbf{B}| - \lambda_{-1}]_- \leq V^2$, and is zero for $2k\hat{\mu}\hat{h}|\mathbf{B}| > \lambda_{-1}$. Therefore, we have

$$\mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_{-1}, \Omega) \lesssim \int_{\Omega\cap\{x:|V_0(x)|>2\}} |V(x)|[|\mathbf{B}(x)| + |V(x)|]dx = 0$$

from assumption (iii). Hence, as $\hbar \to 0$,

$$\begin{aligned} & \tfrac{1}{2}\hat{h}^3\{N(S_1(V_0),\Omega) + N(S_1(-V_0),\Omega)\} - \mathfrak{B}(\hat{\mu}\hat{h}|\mathbf{B}|, -\lambda_1, \Omega) \\ & = O(\epsilon^{\frac{1}{2}} + \eta^{\frac{1}{2}}). \end{aligned} \tag{70}$$

Finally, by [9], Lemma 4.1,

$$\hbar^3 N(S_2, \Omega) \lesssim \int_\Omega [|\mathbf{B}^\infty|^{\frac{3}{2}} + \eta^{-\frac{9}{2}}V_\infty^3]d\mathbf{x} \lesssim \epsilon^{\frac{3}{2}} + \epsilon^3\eta^{-\frac{9}{2}}. \tag{71}$$

It follows from (67) ,(70), and (71) that, with $I_\eta \equiv (-1+\eta, 1-\eta)$,

$$\limsup_{\hbar\to 0}\{\frac{1}{2}\hbar^3[N(\mathbb{D}_V, \Omega, I_\eta) + N(\mathbb{D}_{-V}, \Omega, I_\eta)] - \mathfrak{B}(\mu\hbar|\mathbf{B}|, -\lambda_1, \Omega)\} \lesssim \eta^{\frac{1}{2}}.$$

The reverse inequality is proved similarly, with $\eta < 0$ in (62). The proof is complete. $\square$

The next theorem is the analogue of Theorem 2 in §3.

Theorem 5. *Suppose that*

1. $\mathbf{B}$ *is continuous,*
2. $|\mathbf{B}|,\ V \in L^\infty(\Omega)$,
3. *for some* $p > 3/2$, $V^5,\ b_p^{3/2}V^2 \in L^1(\Omega)$,
4. $\mu\hbar \le$ *constant,*
5. $|\{\mathbf{x} \in \Omega : |V(\mathbf{x})| > 2\}| = 0$.

Then, for all $\gamma \in [0,1]$,

$$\lim_{\eta\to 0}\lim_{\hbar\to 0}\Big\{\frac{\hbar^3}{2}\big[M_\gamma(\mathbb{D}_V,\Omega,I_\eta)+M_\gamma(\mathbb{D}_{-V},\Omega,I_\eta)\big]-\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,-\lambda_1,\Omega)\Big\}=0,$$

where $I_\eta = (-1+\eta, 1-\eta), \eta \in (0,1)$ *and* $\lambda_1 = 2|V| + V^2$.

Proof. The only difference with the proof of Theorem 4 is in the way the error terms involving $S_{\rho r}\cap\Omega$ are estimated to give the analogue of (48). These are now dependent on (46) rather than on the magnetic CLR inequality. We again prove it for $\gamma = 0$. From $(\mathbb{D}_V - V)^2 = (\mathbb{P}_0+1)\mathbb{I}_2$, it readily follows that, for any $\zeta > 0$,

$$\mathbb{D}_V{}^2 \ge \frac{1}{1+\zeta}[\mathbb{P}_0+1] - \frac{1}{\zeta}V^2$$

and, for any $\lambda < 1$, on choosing $\zeta = \lambda/(4-\lambda)$,

$$\mathbb{D}_V{}^2 - 1 + \lambda \ge (3/4)\{\mathbb{P}_0 + \lambda - (4/\lambda)V^2\}.$$

Consequently,

$$\mathbb{R}_{V_0} + \lambda - C\frac{\hbar^2}{(\rho r)^2}\chi_{\rho r} \ge (3/4)\left\{\mathbb{P}_0 + \lambda - (4/\lambda)V_0^2 - C\frac{\hbar^2}{(\rho r)^2}\chi_{\rho r}\right\}$$

and, from (46), with $r = A\hbar$,

$$\begin{aligned}&\hbar^3 N(\mathbb{R}_{V_0}+\lambda - C\tfrac{\hbar^2}{(\rho r)^2}\chi_{\rho r}, S_{\rho r}\cap\Omega)\\ &\lesssim \int_{S_{\rho r}\cap\Omega}\{[V_0^2 + (1/A\rho)^2\chi_{\rho r}]^{5/2} + b_p^{3/2}[V_0^2 + 1/(A\rho)^2\chi_{\rho r}]\}d\mathbf{x}\\ &\lesssim K(\rho r) + (1/A\rho)^5(\rho+\rho r) + (1/A\rho)^2(\rho+\rho r),\end{aligned}$$

as in the proof of Theorem 2, where $K(\rho r)\to 0$ as $\rho \to 0$, uniformly in $\hbar$ and A. This gives (60) as in Theorem 2. From (66) we have,

$$\mathbb{D}_V{}^2 - 1 + 2\lambda + 2\theta \ge (S_1+\lambda) + (S_3+\lambda),$$

where, with $\theta = \eta/(1-\eta)$,

$$S_3 = \eta(1-2\eta)\mathbb{P}_0(\mathbf{B}^\infty) - (1/\theta)V_\infty^2.$$

The $S_1 + \lambda$ term is handled as in the proof of Theorem 2 and the first part above. The S_3 term on $S_{\rho r}\cap\Omega$ is again estimated by (46). □

5. The Pauli Operator: magnetic fields with constant direction.

First, we summarize some facts from [23] that we will need below. Let $\mathbf{B}(\mathbf{x}) = (0,0,B(x_1,x_2))$, $\mathbf{x} = (x, x_3), x = (x_1,x_2)$. Assume that $B = B(x_1,x_2)$ is periodic, i.e., for some T_1, T_2

$$B(x_1,x_2) = B(x_1+T_1,x_2) = B(x_1,x_2+T_2).$$

For $T_3 > 0$ denote by $D^{(3)} = \times_{k=1}^3 [0, T_k)$ the fundamental domain of the lattice Γ with vertices $(T_1 m_1, T_2 m_2, T_3 m_3)$, $m_j \in \mathbb{Z}$, $j = 1, 2, 3$. Choose the following vector potential $\boldsymbol{a}$ corresponding to $\mathbf{B}$: for $\phi = \phi(x_1, x_2)$

$$\boldsymbol{a}(x) = (-\partial_2 \phi, \partial_1 \phi, 0), \qquad \Delta\phi = B.$$

Set

$$\begin{aligned} B_0 = &\ \tfrac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} B(x_1, x_2) dx_1 dx_2, \qquad \mathbf{B}_0 := (0, 0, B_0), \\ B_1 = &\ B - B_0, \\ \phi_0 = &\ \tfrac{B_0}{4}(\alpha x_1^2 + \beta x_2^2), \qquad \alpha + \beta = 2, \\ \phi = &\ \phi_0 + \phi_1, \end{aligned}$$

where $\Delta\phi_1 = B_1$ and ϕ_1 is periodic on $D^{(2)} = [0, T_1) \times [0, T_2)$. Note that the conditions above imply that B_1 is periodic, $\int_0^{T_1} \int_0^{T_2} B_1 dx_1 dx_2 = 0$, and $\Delta\phi_0 = B_0$. The *flux* of B across $D^{(2)}$ is

$$\Phi = \frac{1}{2\pi} \int_{D^{(2)}} B(x_1, x_2) dx_1 dx_2.$$

The physics of our system must be invariant under translations $(x_1, x_2) \to (x_1 + T_1, x_2), (x_1, x_2) \to (x_1, x_2 + T_2)$, since the magnetic field is invariant under these translations. The magnetic potential $\mathbf{a}$ is not invariant, but since $\mathbf{a}$ and the translated $\mathbf{a}, \mathbf{a}_{T_j}, j = 1, 2$, say, give rise to the same magnetic field, there must be gauge transformations

$$(\tau_j u)(\mathbf{x}) := e^{i f_j(\mathbf{x})} u_{T_j}(\mathbf{x}), \qquad j = 1, 2,$$

in which

$$u_{T_1}(\mathbf{x}) = u(x_1 + T_1, x_2), \qquad u_{T_2}(\mathbf{x}) = u(x_1, x_2 + T_2),$$

which are unitary maps on the Hilbert space and

$$(\frac{\hbar}{i}\nabla - \mu\boldsymbol{a})(\tau_j u) = e^{i f_j}(\frac{\hbar}{i}\nabla - \mu\boldsymbol{a}_{T_j}) u_{T_j}, \qquad j = 1, 2.$$

From the discussions above, we see that

$$\boldsymbol{a} = \boldsymbol{a}^0 + \boldsymbol{a}^1, \qquad \boldsymbol{a}^k := (-\partial_2 \phi_k, \partial_1 \phi_k, 0), \qquad k = 0, 1,$$

where $\boldsymbol{a}^1$ is periodic and

$$\begin{aligned} \boldsymbol{a}^0(x_1 + T_1, x_2) = &\ \boldsymbol{a}^0(x_1, x_2) + \tfrac{B_0}{2}\alpha T_1 (0, 1, 0) \\ \boldsymbol{a}^0(x_1, x_2 + T_2) = &\ \boldsymbol{a}^0(x_1, x_2) - \tfrac{B_0}{2}\beta T_2 (1, 0, 0). \end{aligned}$$

The choice $f_1(\mathbf{x}) = -i\frac{\mu\alpha B_0 T_1}{2\hbar} x_2$ and $f_2(\mathbf{x}) = i\frac{\mu\beta B_0 T_2}{2\hbar} x_1$ yield the "magnetic translations" given by

$$\begin{aligned} (\tau_1 u)(x_1, x_2, x_3) = &\ u(x_1 + T_1, x_2, x_3) \exp(-i\mu\alpha B_0 T_1 x_2 / 2\hbar), \\ (\tau_2 u)(x_1, x_2, x_3) = &\ u(x_1, x_2 + T_2, x_3) \exp(i\mu\beta B_0 T_2 x_1 / 2\hbar). \end{aligned}$$

They commute with the expressions $\Pi_k = -i\hbar\partial_k - \mu a_k$, $k = 1, 2, 3$, and $Q_\pm := \Pi_1 \pm i\Pi_2$. If the flux of $\mathbf{B}$ satisfies the condition

$$\mu\hbar^{-1}\Phi = N \in \mathbb{Z} \tag{72}$$

then τ_1 and τ_2 commute and we can reduce our problem to one on the torus $X^{(3)} = \mathbb{R}^3/\Gamma$. In this case, we define the operators $\Pi_k = -i\hbar\partial_k - \mu a_k$, and $Q_\pm$ in $L^2(X^{(3)})$ with domains consisting of functions $u \in C^\infty(\mathbb{R}^3)$ which satisfy

$$\tau_k u = u, \quad k = 1, 2, \qquad u(x_1, x_2, x_3 + T_3) = u(x_1, x_2, x_3).$$

and denote them by $Q_\pm(X^{(3)})$, $\Pi_k(X^{(3)})$. It is proved in [23] that these operators are closable, that each Π_k is symmetric and $Q^*_\pm \subset Q_\mp$. Let

$$\begin{aligned} A_\pm(X^{(3)}) :=& \quad H_0(X^{(3)}) \mp \mu\hbar B \\ \equiv& \quad Q^*_\pm(X^{(3)})Q_\pm(X^{(3)}) + \Pi_3^2(X^{(3)}) \end{aligned}$$

where $H_0 = H_0(\mathbf{B})$, $\mathbf{B} = \nabla \times \boldsymbol{a}$, as defined in §1. We denote the closures of the operators by the same notation. The Pauli operator on the torus $X^{(3)}$, $\mathbb{P}(X^{(3)})$, is now defined by

$$\mathbb{P}_0(X^{(3)}) = \begin{pmatrix} A_+(X^{(3)}) & 0 \\ 0 & A_-(X^{(3)}) \end{pmatrix}$$

and $\mathbb{P}_W(X^{(3)}) = \mathbb{P}_0(X^{(3)}) + W$. We remind the reader that $\mathbb{P}_W(\Omega), \mathbb{P}_W(Q)$ will always stand for the operator $\mathbb{P}_W$ with Dirichlet boundary conditions.

Using a result[3] due to Dubrovin and Novikov [4] that if $\pm N > 0$ (N as defined in (72)) $\lambda = 0$ is an eigenvalue of $A_\pm(X^{(3)})$ of multiplicity $\pm N$, Sobolev proves

Lemma 3. *(Sobolev* [23]*, Lemma 4.3, $\gamma = 0$, $d = 3$) Let (72) be satisfied and $|B| \geq \kappa > 0$. If W is constant in $D^{(3)}$ and $W_- < 2\mu\hbar\kappa$, then*

$$\left|N(\mathbb{P}_W, X^{(3)}) - \frac{T_3}{\pi\hbar} W_-^{\frac{1}{2}} |N|\right| \leq |N|.$$

We shall follow Sobolev's strategy, which is again basically inspired by the method of Colin de Verdiere [3], but substantially modified to meet the needs of the constant direction magnetic field case, using the observations and results noted above. There will now be an infinity of negative eigenvalues in general, and so we can only expect results for $N(\mathbb{P}_W + \lambda, \Omega)$ with $\lambda > 0$; in fact for $M_\gamma(\mathbb{P}_W + \lambda, \Omega)$, $\lambda > 0$ when $\gamma \in [0, 1/2]$. When $\gamma > 1/2$, $M_\gamma(\mathbb{P}_W, \Omega)$ is finite and its semi-classical limit is given by Sobolev in [23]. Our analysis requires an estimate for $\hbar^3 N(\mathbb{P}_W + \lambda, \Omega), \lambda > 0$, in which $\mu|B|$ occurs linearly. A suitable result is obtained by Shen in [20]. In it $N(\mathbb{P}_W + \lambda, \Omega)$ grows like $\lambda^{-1/2}$ as $\lambda \to 0+$, and we are forced to take $\lambda > 0$ in our result for $M_\gamma(\mathbb{P}_W + \lambda, \Omega)$ when $\gamma \leq 1/2$ in view of (26). If Ω is a bounded domain we prove in the next lemma that the λ dependence in $N(\mathbb{P}_W + \lambda, \Omega)$ is at most logarithmic; of course, this is still crude since the spectrum of $\mathbb{P}_W(\Omega)$ is discrete in this case, but it does have interesting implications for $M_\gamma(\mathbb{P}_W, \Omega)$ for $\gamma > 0$, and, moreover, the technique used to prove it can be made to yield a similar estimate for unbounded domains; see Proposition 5 below.

Lemma 4. *Let $Q = Q^{(2)} \times [0, R)$, $W_- \in L^\infty(Q)$, and $\mathbf{B} = (0, 0, B)$. Then, for $\lambda > 0$ and $\tilde{W} := \inf_Q W(x)$*

$$N(\mathbb{P}_W + \lambda, Q) \lesssim [(\tilde{W}_- + 1)e^{\tilde{W}_-} |\log \lambda^{-1}|]\mu\hbar^{-2}(\tilde{W} + \lambda)_-^{\frac{1}{2}} |Q| \max_{Q^{(2)}} |B(x)|. \tag{73}$$

Proof. We have that

$$\mathbb{P}_W = \mathbb{P}_0^{(2)} + W - \hbar^2\partial_3^2 \geq \mathbb{P}_0^{(2)} + \tilde{W} - \hbar^2\partial_3^2 = \mathbb{P}_{\tilde{W}}$$

where $\mathbb{P}_0^{(2)}$ is the Pauli operator in two dimensions, namely

$$\mathbb{P}_0^{(2)} = \begin{pmatrix} A_+^{(2)} & 0 \\ 0 & A_-^{(2)} \end{pmatrix}$$

[3]See also Appendix A of Sobolev [23] for a proof.

with

$$A_{\pm}^{(2)} = Q_{\pm}^{*}Q_{\pm} = H_0^{(2)} \mp \mu\hbar B$$

and

$$H_0^{(2)} = \sum_{k=1}^{2}(\frac{\hbar}{i}\partial_k - \mu a_k)^2$$

the magnetic Schrödinger operator in $\mathbb{R}^2$. The eigenvalues of $-\hbar^2\partial_3^2$ with Dirichlet boundary conditions on $(0,R)$ are $\epsilon_m = (m\pi\hbar)^2/R^2$, $m \in \mathbb{N}$. It follows that the eigenvalues of $\mathbb{P}_{\tilde{W}}$ are of the form $\lambda_n(\mathbb{P}_0^{(2)} + \tilde{W}) + \epsilon_m$, where $\lambda_n(\mathbb{P}_0^{(2)} + \tilde{W})$ are the eigenvalues of $\mathbb{P}_0^{(2)} + \tilde{W}$ on $Q^{(2)}$ (with Dirichlet boundary conditions). Hence, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned} N(\mathbb{P}_W + \lambda, Q) \leq & \sum_m N(\mathbb{P}_0^{(2)} + \tilde{W} + \epsilon_m + \lambda, Q^{(2)}) \\ = & \sum_m \sum_{j=1}^{2} N(H_0^{(2)} + (-1)^j \mu\hbar B + \tilde{W} + \epsilon_m + \lambda, Q^{(2)}). \end{aligned} \tag{74}$$

Since $\mathbb{P}_0^{(2)} \geq 0$, we have $A_{\pm}^{(2)} = H_0^{(2)} \mp \mu\hbar B \geq 0$. Hence, for any $\epsilon \in (0,1)$

$$\begin{aligned} H_0^{(2)} \mp \mu\hbar B + \tilde{W} + \epsilon_m + \lambda \geq & \ \epsilon(H_0^{(2)} \mp \mu\hbar B) + \tilde{W} + \epsilon_m + \lambda \\ \geq & \ \epsilon H_0^{(2)} - \epsilon\mu\hbar|B| + \tilde{W} + \epsilon_m + \lambda. \end{aligned}$$

Replacing $A_{\pm}$ by something smaller seems rather wasteful, but will be seen to provide the correct scaling for $\mu|B|$ by a suitable choice of ε. It is reminiscent of a technique (running energy-scale renormalization) used in a paper of Lieb, Loss and Solovej [14]. On substituting in (74), this yields

$$N(\mathbb{P}_W + \lambda, Q) \leq 2\sum_m N(\epsilon H_0^{(2)} - \epsilon\mu\hbar|B| + \tilde{W} + \epsilon_m + \lambda, Q^{(2)}) \tag{75}$$

where the sum is over all m such that

$$\epsilon_m = \frac{(m\pi\hbar)^2}{R^2} < (\tilde{W} + \lambda)_-. \tag{76}$$

To estimate $N(\epsilon H_0^{(2)} - \epsilon\mu\hbar|B| - \tilde{W}_- + \epsilon_m + \lambda, Q^{(2)})$ we use the result of Rozenblum and Solomyak [[19], Theorem 2.4] with (in their notation)

$$\mathcal{A} = \epsilon H_0^{(2)} + \epsilon_m + \lambda, \quad \mathcal{B} = -\epsilon\hbar^2\Delta + \epsilon_m + \lambda, \quad V = (\epsilon\mu\hbar|B| + \tilde{W}_-)\chi_{Q^{(2)}}.$$

Then, $\mathcal{B} \in \mathcal{P}$ as it generates a positivity preserving contractive semigroup with the $(2,\infty)$-boundedness (ultracontractivity) property having kernel $Q(t,x,y)$, where, on the diagonal,

$$Q(t,x,x) = \frac{1}{2\pi\hbar^2\epsilon t}e^{-(\epsilon_m+\lambda)t};$$

see [19], §2, for the terminology. The operator $\mathcal{A}$ is dominated by $\mathcal{B}$ in the sense that

$$|e^{-t\mathcal{A}}\psi(x)| \leq e^{-t\mathcal{B}}|\psi(x)| \quad \text{for a.e. } x \in \mathbb{R}^2, \quad x = (x_1, x_2),$$

i.e., $\mathcal{A} \in \mathcal{PD}(\mathcal{B})$ in the language of [19], §2.3. It follows from [[19], Theorem 2.4] with the choice $G(z) = (z-k)_+$ for some $k > 0$, that

$$\begin{aligned} N(\mathcal{A} - V) \leq & \ \frac{1}{g(1)}\int_0^\infty \frac{dt}{t}\int_{Q^{(2)}} \frac{1}{2\pi\hbar^2\epsilon t}e^{-(\epsilon_m+\lambda)t}(tV(x) - k)_+ dx \\ = & \ \frac{1}{2\pi g(1)}\int_{Q^{(2)}} dx \int_{k/V(x)}^\infty e^{-(\epsilon_m+\lambda)t}(tV(x) - k)\frac{dt}{\epsilon\hbar^2 t^2} \\ = & \ \frac{1}{2\pi\hbar^2\epsilon g(1)}\int_{Q^{(2)}} V(x)dx \int_{\phi(x)}^\infty e^{-(\epsilon_m+\lambda)t}(t - \phi(x))\frac{dt}{t^2} \end{aligned} \tag{77}$$

for $\phi(x) := k/V(x)$ and

$$g(1) = \int_0^\infty (z-k)_+ e^{-z} \frac{dz}{z} = \int_1^\infty e^{-ks} \frac{ds}{s^2}. \tag{78}$$

Now choose $\epsilon = \tilde{W}_-/(\mu\hbar \max_{Q^{(2)}} |B(x)|)$, $k = 2\tilde{W}_-$. Then $\phi(x) \geq 1$ and

$$\int_{\phi(x)}^\infty e^{-(\epsilon_m+\lambda)t}(t-\phi(x))\frac{dt}{t^2} \leq \int_1^\infty e^{-\lambda t}(t-1)\frac{dt}{t^2} \leq \int_1^\infty e^{-\lambda t}\frac{dt}{t} \leq |\log\lambda^{-1}| + O(1)$$

as $\lambda \to 0$. Furthermore, on integration by parts,

$$g(1) = \frac{e^{-k}}{k} - \frac{2}{k}\int_1^\infty e^{-ks}\frac{ds}{s^3} \geq \frac{e^{-k}}{k} - \frac{2}{k}g(1)$$

implying that

$$g(1) \geq e^{-k}/(k+2). \tag{79}$$

It now follows from (75), (76), (77), and (79) that

$$N(\mathbb{P}_W + \lambda, Q) \lesssim (\tilde{W}+\lambda)_-^{\frac{1}{2}} \frac{R}{\pi\hbar}(k+2)e^k(|\log\lambda^{-1}|)\mu\hbar^{-1}|Q^{(2)}| \max_{Q^{(2)}} |B(x)|$$

and (73) is proved. □

The method of proving Lemma 4 also yields Proposition 5 at the end of this section for an unbounded Ω, and an operator $\mathbb{P}_W(\Omega)$ with a non-empty essential spectrum. However, we leave the result till then so as not to break the flow of the argument leading to the proof of Theorem 6.

Lemma 5. *Let $Q = [0,R)^3$, W constant on Q, and $|B| \geq \kappa > 0$. Then, for any $\lambda > 0$,*

$$\lim_{\hbar\to 0}\{\mu^{-1}\hbar^2 N(\mathbb{P}_W+\lambda, Q) - \frac{1}{2\pi^2}\int_Q |B|(W+\lambda)_-^{\frac{1}{2}} d\mathbf{x}\} = 0 \tag{80}$$

with $2\mu\hbar\kappa \geq W_-$. For any $\gamma > 0$

$$\lim_{\hbar\to 0}\{\mu^{-1}\hbar^2 M_\gamma(\mathbb{P}_W, Q) - \beta_\gamma \int_Q |B| W_-^{\gamma+\frac{1}{2}} d\mathbf{x}\} = 0 \tag{81}$$

with $2\mu\hbar\kappa \geq W_-$, in which

$$\beta_\gamma := \frac{1}{4\pi^2}\int_0^1 t^\gamma (1-t)^{-\frac{1}{2}} dt.$$

Proof. As in [[23], Lemma 5.1], let $\Omega_j^{(2)}$, $j \in \mathbb{N}$, be non-overlapping squares obtained by translating $Q^{(2)} = [0,R)^2$ parallel to the coordinate axes to form a tesselation of $\mathbb{R}^2$. Choose a square $D^{(2)} \subset \mathbb{R}^2$ such that

$$\mu\hbar^{-1}\frac{1}{2\pi}\int_{D^{(2)}} B(x)dx = N \in \mathbb{Z}$$

with B extended by periodicity from $Q^{(2)}$ to $\mathbb{R}^2$. Let $M = \#\{j : \Omega_j^{(2)} \subset D^{(2)}\}$ and suppose, without loss of generality, that

$$(1-\epsilon)|N| \leq \mu\hbar^{-1}M|\Phi| \leq |N|, \qquad \Phi = \frac{1}{2\pi}\int_{Q^{(2)}} B(x)dx \tag{82}$$

for $\epsilon \in (0, \frac{1}{2})$. Set $D^{(3)} = D^{(2)} \times [0, R)$ and denote by $X^{(d)}$ the torus with fundamental domain $D^{(d)}$, $d = 2, 3$. It is proved in [[23], (5.3)] that if $[W + \lambda]_- < 2\mu\hbar\kappa$

$$\limsup_{\hbar\to 0} \mu^{-1}\hbar^2 N(\mathbb{P}_W + \lambda, Q) \le \frac{1}{2\pi^2}\int_Q |B|(W+\lambda)_-^{\frac{1}{2}} d\mathbf{x}, \qquad \forall \lambda \ge 0. \tag{83}$$

For the lower bound we need $\lambda > 0$ in order to use Lemma 4. Let

$$\begin{aligned} S^{(2)} = &\quad \overline{X^{(2)}} \setminus \cup_{j=1}^M int(\Omega_j^{(2)}) \\ S_\delta^{(2)} = &\quad \{x \in X^{(3)} :\ dist(x, S^{(2)}) < \delta\} \end{aligned}$$

for $\delta \in (0, \frac{R}{2})$, and set $S^{(3)} = S^{(2)} \times [0, R)$, $S_\delta^{(3)} = S_\delta^{(2)} \times [0, R)$. Then, $X^{(d)} \subseteq S_\delta^{(d)} \cup_{j=1}^M int(\Omega_j^{(d)})$ for $d = 2, 3$. Also, since $|B| \ge \kappa$

$$\begin{aligned} \tfrac{1}{2\pi}\kappa|S^{(2)}| \le &\quad \tfrac{1}{2\pi}\textstyle\int_{S^{(2)}} |B|dx \\ = &\quad \tfrac{1}{2\pi}\textstyle\int_{D^{(2)}} |B|dx - \tfrac{1}{2\pi}M\int_{\Omega_j^{(2)}} |B|dx \\ \le &\quad \tfrac{\epsilon|N|}{\mu\hbar^{-1}} \end{aligned}$$

from (82); note that B is of one sign since B is continuous and $|B| \ge \kappa$. Therefore,

$$|S^{(2)}| \le \frac{2\pi}{\kappa\mu\hbar^{-1}}\epsilon|N|$$

and

$$|S_\delta^{(2)}| \lesssim \frac{\mu^{-1}\hbar\epsilon}{\kappa}|N| + M\delta$$

which implies that

$$\begin{aligned} |S_\delta^{(3)}| \lesssim &\quad \tfrac{\mu^{-1}\hbar\epsilon}{\kappa}|N|R + MR\delta \\ \le &\quad \tfrac{M\epsilon}{\kappa(1-\epsilon)}|\Phi|R + MR\delta \\ \le &\quad C(\epsilon+\delta)MR \end{aligned} \tag{84}$$

where C depends on κ and Φ. We now proceed as in (33). The analogue is

$$\begin{aligned} ([\mathbb{P}_{W+\lambda}(X^{(3)}) + \tfrac{C\hbar^2}{\delta^2}]f, f) \ge &\quad (\mathbb{P}_{W+\lambda}(S_\delta^{(3)})\psi_0 f, \psi_0 f) \\ &\quad + \textstyle\sum_{j=1}^M (\mathbb{P}_{W+\lambda}(\Omega_j^{(3)})\psi_j f, \psi_j f) \end{aligned}$$

for f in the domain of $\mathbb{P}_W(X^{(3)})$ and a partition of unity $\{\psi_j\}_0^M$ subordinate to the covering of $X^{(3)}$ by $S_\delta^{(3)} \cup_{j=1}^M \Omega_j^{(3)}$ which satisfies (31) with $K = M$ and δ replacing ρr. From this we conclude that

$$N(\mathbb{P}_W + \lambda + \tfrac{C\hbar^2}{\delta^2}, X^{(3)}) \le N(\mathbb{P}_W + \lambda, S_\delta^{(3)}) + MN(\mathbb{P}_W + \lambda, Q) \tag{85}$$

From Lemma 3

$$N(\mathbb{P}_W + \lambda + \frac{C\hbar^2}{\delta^2}, X^{(3)}) \ge \frac{1}{\pi\hbar}R(W + \lambda + \frac{C\hbar^2}{\delta^2})_-^{\frac{1}{2}}|N| - |N| \tag{86}$$

and by Lemma 4 and (84)

$$N(\mathbb{P}_W + \lambda, S_\delta^{(3)}) \lesssim \mu\hbar^{-2}|\log\lambda^{-1}||S_\delta^{(3)}| \lesssim \mu\hbar^{-2}|\log\lambda^{-1}|(\epsilon+\delta)MR. \tag{87}$$

It follows from (85)-(87) that

$$MN(\mathbb{P}_W + \lambda, Q) \ge \frac{1}{\pi\hbar}R(W + \lambda + \frac{C\hbar^2}{\delta^2})_-^{\frac{1}{2}}|N| - |N| - C\mu\hbar^{-2}|\log\lambda^{-1}|(\epsilon+\delta)RM.$$

Since

$$|(W + \lambda + \frac{C\hbar^2}{\delta^2})_-^{\frac{1}{2}} - (W+\lambda)_-^{\frac{1}{2}}| \lesssim \hbar/\delta$$

we have from (82)

$$\mu^{-1}\hbar^2 N(\mathbb{P}_W+\lambda,Q)\geq \frac{R}{\pi}(W+\lambda)_-^{\frac{1}{2}}|\Phi| - C(\frac{\hbar}{\delta}R|\Phi|+\hbar|\Phi|+|\log\lambda^{-1}|(\epsilon+\delta)R)$$

which yields (80) on recalling (83). Note that (87) gives, for any $\gamma>0$,

$$M_\gamma(\mathbb{P}_W,S_\delta^{(3)})=\gamma\int_0^\infty \lambda^{\gamma-1}N(\mathbb{P}_W+\lambda,S_\delta^{(3)})d\lambda \lesssim \mu\hbar^{-2}(\epsilon+\delta)RM. \tag{88}$$

A similar argument then gives (81). □

For $|B|\geq\kappa$ and $2\mu\hbar\kappa\geq W_-$ we have $2k\mu\hbar|B|+W\geq 2\mu\hbar\kappa-W_-\geq 0$, for $k\geq 1$, and, for any $\gamma\geq 0$,

$$\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,W,Q)=\mu\hbar\beta_\gamma\int_Q|B|W_-^{\gamma+\frac{1}{2}}d\mathbf{x}.$$

This fact gives

Corollary 1. *Let $Q=[0,R)^3$, W a constant on Q, and $|B|\geq\kappa>0$. Then, for any $\lambda>0$,*

$$\lim_{\hbar\to 0}\frac{\hbar^3}{\mu\hbar+1}\{N(\mathbb{P}_W+\lambda,Q)-\hbar^{-3}\mathfrak{B}_0(\mu\hbar|\mathbf{B}|,W+\lambda,Q)\}=0 \tag{89}$$

uniformly in $\mu\geq 0$. Moreover, for any $\gamma>0$

$$\lim_{\hbar\to 0}\frac{\hbar^3}{\mu\hbar+1}\{M_\gamma(\mathbb{P}_W,Q)-\hbar^{-3}\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,W,Q)\}=0.$$

In order to remove the assumption that $|B|\geq\kappa$, we prove the following

Lemma 6. *Let $Q=[0,R)^3$ and W a nonzero constant on Q. Then, for $\lambda>0$*

$$\begin{array}{rl}\limsup_{\hbar\to 0}\frac{\hbar^3}{\mu\hbar+1} & |N(\mathbb{P}_W+\lambda,Q)-\hbar^{-3}\mathfrak{B}_0(\mu\hbar|\mathbf{B}|,W+\lambda,Q)| \\ \leq & C|\log\lambda^{-1}|W_-^{\frac{1}{2}}|Q|\max_{Q^{(2)}}|B(x)|,\end{array} \tag{90}$$

and for any $\gamma>0$

$$\begin{array}{rl}\limsup_{\hbar\to 0}\frac{\hbar^3}{\mu\hbar+1} & |M_\gamma(\mathbb{P}_W,Q)-\hbar^{-3}\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,W+\lambda,Q)| \\ \leq & CW_-^{\gamma+\frac{1}{2}}[|\log W_-|+1]|Q|\max_{Q^{(2)}}|B(x)|.\end{array} \tag{91}$$

Both (90) and (91) hold uniformly in $\mu\geq 0$.

Proof. By Theorem 1

$$\lim_{\hbar\to 0}\sup_{\mu\hbar\leq C}\frac{\hbar^3}{\mu\hbar+1}|N(\mathbb{P}_W+\lambda,Q)-\mathfrak{B}(\mu\hbar|\mathbf{B}|,W+\lambda,Q)|=0. \tag{92}$$

From Lemma 4

$$\frac{\hbar^3}{\mu\hbar+1}N(\mathbb{P}_W+\lambda,Q)\lesssim|\log\lambda^{-1}|(W+\lambda)_-^{\frac{1}{2}}|Q|\max_{Q^{(2)}}|B(x)|$$

and

$$\begin{array}{l}\frac{\hbar^3}{\mu\hbar+1}\hbar^{-3}\mathfrak{B}_0(\mu\hbar|B|,W+\lambda,Q) \\ \quad\leq\int_Q|B|[(W+\lambda)_-^{\frac{1}{2}}+2\sum_{k\geq 1}(2k\mu\hbar|B|+W+\lambda)_-^{\frac{1}{2}}]d\mathbf{x} \\ \quad\lesssim\int_Q(W+\lambda)_-^{\frac{1}{2}}[|B|+(\mu\hbar)^{-1}(W+\lambda)_-]d\mathbf{x}.\end{array}$$

Therefore,

$$\sup_{\mu\hbar\geq C}\tfrac{\hbar^3}{\mu\hbar+1}|N(\mathbb{P}_W+\lambda,Q)-\hbar^{-3}\mathfrak{B}(\mu\hbar|\mathbf{B}|,W+\lambda,Q)| \\ \lesssim|\log\lambda^{-1}|(W+\lambda)_-^{\frac{1}{2}}|Q|\max_{Q^{(2)}}|B(x)|+\tfrac{1}{C}|Q|(W+\lambda)_-^{\frac{3}{2}}$$

and (90) follows since C is arbitrary. The proof of (91) follows in a similar manner. □

Theorem 6. *Let* $\mathbf{B}(\mathbf{x})=(0,0,B(x))$. *Suppose that*

1. *B is continuous,*
2. *$W_-\in L^\infty(\Omega)$,*
3. *for some $p>1, W_-^2, \hat{b}_pW_-\in L^1(\Omega)$, where $\hat{b}_p$ is defined in (10).*

Then, for all $\gamma\geq 0$ and $\lambda>0$,

$$\lim_{\hbar\to 0}\frac{\hbar^3}{\mu\hbar+1}\left\{M_\gamma(\mathbb{P}_W+\lambda,\Omega)-\hbar^{-3}\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,W+\lambda,\Omega)\right\}=0. \tag{93}$$

uniformly for $\mu\geq 0$. If $\gamma>1/2$, (93) holds for $\lambda=0$.

Proof. We prove the result for $\gamma=0$, the proof for $\gamma>0$ being similar. The last assertion in the theorem, concerning $\gamma>1/2$, is proved in [23]. Let W_0, Q_k, $k=1,2,\dots,k$ be as in §2 (A_1)-(A_3). As in §6 of Sobolev [23], for $\eta>0$, we partition each Q_k into a finite number of cubes Q_{jk}, $j=1,2,\dots,$ such that for each pair k,j

$$|B(x)-B(y)|\leq\eta,\qquad x,y\in Q_{k,j}. \tag{94}$$

Let $I_+\equiv I_+(\eta):=\{(k,j):\max_{Q_{k,j}}|B(x)|\geq 2\eta\}$ and $I_-\equiv I_-(\eta)$ the complementary set. In view of (94) we have that $|B(x)|\geq\eta$ for $x\in Q_{k,j}$, $(k,j)\in I_+$. Since $\mathbb{P}_{W_0}(\Omega)\leq\oplus_{k=1}^K\mathbb{P}_{W_0}(Q_k)$,

$$N(\mathbb{P}_{W_0}+\lambda,\Omega)\geq\sum_{k=1}^K N(\mathbb{P}_{W_0}+\lambda,Q_k)$$

and from Corollary 1 and Lemma 6

$$\liminf_{\hbar\to 0}\tfrac{\hbar^3}{\mu\hbar+1}\left\{N(\mathbb{P}_{W_0}+\lambda,\Omega)-\hbar^{-3}\mathfrak{B}_0(\mu\hbar|\mathbf{B}|,W_0+\lambda,\Omega)\right\} \\ \geq -C\eta\sum_{(k,j)\in I_-}\int_{Q_{k,j}}(W_0+\lambda)_-^{\frac{1}{2}}dx\gtrsim-\eta.$$

Since η is arbitrary, then

$$\liminf_{\hbar\to 0}\frac{\hbar^3}{\mu\hbar+1}\left\{N(\mathbb{P}_{W_0}+\lambda,\Omega)-\hbar^{-3}\mathfrak{B}_0(\mu\hbar|\mathbf{B}|,W_0+\lambda,\Omega)\right\}\geq 0 \tag{95}$$

for each $\lambda>0$. For the reverse inequality we proceed as in the proof of (37). Let the interior of each Q_k be denoted by $int(Q_k)$. Set

$$\begin{aligned} &S:=\Omega\setminus\cup_{k=1}^K int(Q_k),\qquad S_{\rho r}:=\{x\in\Omega:dist(x,S)<\rho r\},\\ &Q(\rho r):=\{x\in\Omega:dist(x,Q)<\rho r\}\end{aligned} \tag{96}$$

for some $\rho \in (0,1)$. Construct a partition of unity $\{\psi_k\}_{k=-1}^K$ subordinate to the covering $\cup_{k=1}^K int(Q_k) \cup S_{\rho r}$ of Ω:

(97)
$$\begin{array}{rl} (i) & \psi_{-1} \in C^\infty(\Omega), \quad \psi_0 \in C_0^\infty(S_{\rho r} \cap Q_{\rho r}), \quad \psi_k \in C_0^\infty(int(Q_k)), \ k = 1, \dots, K; \\ (ii) & \sum_{k=-1}^K \psi_k^2 \equiv 1, \forall x; \\ (iii) & \psi_{-1}(x) = 0, \ x \in Q, \quad \psi_{-1}(x) = 1, \ x \notin Q(\rho r); \\ (iv) & \sum_{k=-1}^K |\nabla \psi_k(x)|^2 \le C(\rho r)^{-2} \chi_{S_{\rho r} \cap Q(\rho r)} \end{array}$$

for some $C > 0$ where $\chi_{\rho r}$ is the characteristic function for $S_{\rho r} \cap Q(\rho r)$. It follows that

$$|S_{\rho r} \cap Q| \lesssim \rho |Q|, \qquad |Q(\rho r) \setminus Q| \lesssim \rho r, \tag{98}$$

and for every $f \in [C_0^\infty(\Omega)]^2$

$$\sum_{k=-1}^K (\mathbb{P}_0 \psi_k f, \psi_k f) \le (\mathbb{P}_0 f, f) + C\hbar^2 (\rho r)^{-2} (\chi_{S_{\rho r} \cap Q(\rho r)} f, f).$$

Let $r = A\hbar$ and $\lambda - C(A\rho)^{-2} \ge 0$. Then,

$$\begin{array}{rl} ((\mathbb{P}_{W_0} + \lambda) f, f) \ge & \sum_{k=-1}^K ([\mathbb{P}_{W_0} + \lambda - C\hbar^2 (\rho r)^{-2} \chi_{\rho r}] \psi_k f, \psi_k f) \\ \ge & \sum_{k=0}^K ([\mathbb{P}_{W_0} + \lambda - C\hbar^2 (\rho r)^{-2} \chi_{\rho r}] \psi_k f, \psi_k f) \end{array} \tag{99}$$

since $\psi_{-1} \equiv 0$ in Q,and $W_0 = 0$ for $x \notin Q$. This implies that

$$\begin{array}{rl} N(\mathbb{P}_{W_0} + \lambda, \Omega) \le & N(\mathbb{P}_{W_0} + \lambda - C\hbar^2 (\rho r)^{-2}, S_{\rho r} \cap Q(\rho r)) \\ & + \sum_{k=1}^K N(\mathbb{P}_{W_0} + \lambda - C\hbar^2 (\rho r)^{-2}, Q_k). \end{array} \tag{100}$$

From Corollary 1 and Lemma 6

$$\begin{array}{rl} \limsup_{\hbar \to 0} \frac{\hbar^3}{\mu\hbar + 1} \sum_{k=1}^K & \{N(\mathbb{P}_{W_0} + \lambda - \frac{C\hbar^2}{(\rho r)^2}, Q_k) \\ & -\hbar^{-3} \mathfrak{B}_0(\mu\hbar |\mathbf{B}|, W_0 + \lambda - \frac{C\hbar^2}{(\rho r)^2}, Q_k)\} \\ \le & C\eta \sum_{(k,j) \in I_-} \int_{Q_{k,j}} (W_0 + \lambda)_-^{\frac{1}{2}} d\mathbf{x} \lesssim \eta. \end{array} \tag{101}$$

and from (22),

$$\begin{array}{l} \frac{1}{\mu\hbar + 1} |\mathfrak{B}_0(\mu\hbar |\mathbf{B}|, W_0 + \lambda - \frac{C\hbar^2}{(\rho r)^2}, Q_k) - \mathfrak{B}_0(\mu\hbar |\mathbf{B}|, W_0 + \lambda, Q_k)| \\ \lesssim \frac{1}{A\rho} + \frac{1}{(A\rho)^3}. \end{array} \tag{102}$$

From (73) and (98)

$$\begin{array}{l} \mu^{-1} \hbar^2 N(\mathbb{P}_{W_0} + \lambda - C\hbar^2 (\rho r)^{-2}, S_{\rho r} \cap Q(\rho r)) \\ \lesssim e^{1/(A\rho)^2} \frac{1}{(A\rho)^3} \rho |Q|. \end{array} \tag{103}$$

It now follows from (100)-(103), on allowing $\hbar \to 0, A \to \infty, \eta \to 0, \rho \to 0$, in that order, that

$$\limsup_{\hbar \to 0} \frac{\hbar^3}{\mu\hbar + 1} \{N(\mathbb{P}_{W_0} + \lambda, \Omega) - \hbar^{-3} \mathfrak{B}_0(\mu\hbar |\mathbf{B}|, W_0 + \lambda, \Omega)\} \le 0$$

which together with (95) implies

$$\lim_{\hbar \to 0} \frac{\hbar^3}{\mu\hbar + 1} \{N(\mathbb{P}_{W_0} + \lambda, \Omega) - \hbar^{-3} \mathfrak{B}_0(\mu\hbar |\mathbf{B}|, W_0 + \lambda, \Omega)\} = 0. \tag{104}$$

Similarly, we have for any $\gamma > 0$ and $\lambda > 0$

$$\lim_{\hbar\to 0}\frac{\hbar^3}{\mu\hbar+1}\{M_\gamma(\mathbb{P}_{W_0}+\lambda,\Omega)-\hbar^{-3}\mathfrak{B}_\gamma(\mu\hbar|\mathbf{B}|,W_0+\lambda,\Omega)\}=0. \tag{105}$$

The proof for general W is similar to the corresponding part of the proof of Theorem 2 in §3, using the Weyl inequalities in the same way as Sobolev does in [23], but, instead of (46) we now use the inequality derived by Shen in Lemmas 5.1, 5.3, 5.4 and (5.14) of [20], namely,

$$\hbar^3 N(\mathbb{P}_W+\lambda,\mathbb{R}^3)\lesssim\frac{1}{\sqrt{\lambda}}\Big\{\int_{\mathbb{R}^3}|W_-(\mathbf{x})|^2d\mathbf{x}+\mu\hbar\int_{\mathbb{R}^3}\hat{b}_p(\mathbf{x})W_-(\mathbf{x})d\mathbf{x}\Big\}. \tag{106}$$

□

The following analogue of Theorem 3 is also readily proved.

Theorem 7. *Let* $\mathbf{B}(\mathbf{x})=(0,0,B(x))$. *Suppose that*

(i) W, $\mathbf{B}$ *are continuous,*
(ii) $|W(\mathbf{x})|\to 0$ *uniformly as* $|\mathbf{x}|\to\infty$ *in* Ω.

Then, (93) holds for all $\gamma\in[0,1)$ *and* $\lambda>0$.

Finally in this section we give the result promised after Lemma 4.

Proposition 5. *Let* $\Omega=\Omega^{(2)}\times\mathbb{R}$, *where* $\Omega^{(2)}$ *is of finite measure in* $\mathbb{R}^2$, *and suppose that*

(i) $\mathbf{B}=(0,0,B(x))$ *and* $|\mathbf{B}|\in L^\infty(\Omega^{(2)})$;
(ii) $\inf_\Omega W(\mathbf{x})\geq -c\chi_{(-R,R)}(x_3)$, $\mathbf{x}=(x,x_3)$, *for some positive numbers* c, R.

Then for $\lambda>0$

$$N(\mathbb{P}_W+\lambda,\Omega)\lesssim\hbar^{-3}(\mu\hbar+1)\sqrt{c}|\log\lambda^{-1}|R\int_{\Omega^{(2)}}(|B(x)|+c)dx \tag{107}$$

and, for any $\gamma>0$,

$$M_\gamma(\mathbb{P}_W,\Omega)\lesssim\hbar^{-3}(\mu\hbar+1)c^{\gamma+\frac{1}{2}}(|\log c|+1)R\int_{\Omega^{(2)}}(|B(x)|+c)dx. \tag{108}$$

Proof. In the notation of the proof of Lemma 4,

$$\mathbb{P}_W\geq\mathbb{P}_0^{(2)}+(-\hbar^2\partial_3^2-\tilde{W})$$

where $\tilde{W}(\mathbf{x})=\tilde{W}(x_3)=c\chi_{(-R,R)}(x_3)$. The expression $-\hbar^2\partial_3^2-\tilde{W}$ has a self-adjoint realization in $L^2(\mathbb{R})$ with essential spectrum $[0,\infty)$ (see [18], Proposition 2.1), and it is readily shown that there are negative eigenvalues at the solutions of the equation

$$\tan\Big(\frac{2R}{\hbar}\sqrt{c-\lambda}\Big)=4\frac{\sqrt{\lambda(c-\lambda)}}{c-2\lambda},\qquad\lambda\in(0,c) \tag{109}$$

corresponding to the eigenfunctions

(110)

$$\varphi(x_3,\lambda)=\begin{cases}\frac{e^{2R\sqrt{\lambda}/\hbar}}{2i\theta}\big[(1+i\theta)e^{(1+i\theta)\frac{R\sqrt{\lambda}}{\hbar}-i\theta\frac{x_3\sqrt{\lambda}}{\hbar}}-(1-i\theta)e^{(1-i\theta)\frac{R\sqrt{\lambda}}{\hbar}+i\theta\frac{x_3\sqrt{\lambda}}{\hbar}}\big], & |x_3|\leq R,\\ e^{-\frac{\sqrt{\lambda}}{\hbar}x_3}, & x_3>R,\\ e^{\frac{\sqrt{\lambda}}{\hbar}x_3}, & x_3<-R,\end{cases}$$

where $\theta := \sqrt{\frac{c}{\lambda} - 1}$. If these negative eigenvalues are denoted by $-\epsilon_m$, $m = 1, \ldots, M$, then $\epsilon_m \leq c$ and

$$M \leq \#\{n : c - \frac{(n\pi\hbar)^2}{4R^2} \geq 0\} = \left[2\frac{R\sqrt{c}}{\pi\hbar}\right]. \tag{111}$$

We now proceed in a similar way to the proof of Lemma 4. We have that for any $\epsilon \in (0,1)$

$$\begin{aligned} N(\mathbb{P}_W + \lambda, \Omega) \leq & \sum_{m=1}^{M} N(\mathbb{P}_0^{(2)} + \lambda - \epsilon_m, \Omega^{(2)}) \\ \leq & \ 2\sum_{m=1}^{M} N(\epsilon H_0^{(2)} - \epsilon\mu\hbar|B| + \lambda - \epsilon_m, \Omega^{(2)}). \end{aligned} \tag{112}$$

Now substitute

$$\mathcal{A} = \epsilon H_0^{(2)} + \lambda, \quad \mathcal{B} = -\epsilon\hbar^2\Delta + \lambda, \quad V = (\epsilon\mu\hbar|B| + \epsilon_m)\chi_{\Omega^{(2)}}$$

in Theorem 2.4 of [19]. This gives, as in (77),

$$N(\mathcal{A} - V) \leq \frac{1}{2\pi\hbar^2\epsilon g(1)} \int_{\Omega^{(2)}} V(x)dx \int_{\phi(x)}^{\infty} e^{-\lambda t}(t - \phi(x))\frac{dt}{t^2}$$

where $\phi(x) = k/V(x)$ and $g(1)$ satisfies (78). Choose

$$\epsilon = \frac{1}{\mu\hbar + 1}, \quad k = \max_{\Omega^{(2)}} |B(x)| + c.$$

Then $\phi(x) \geq 1$ and the result follows as before. □

6. The Dirac operator: magnetic fields with constant direction.

Theorem 8. *Suppose that*

1. *B is continuous,*
2. *$V_- \in L^\infty(\Omega)$,*
3. *for some $p > 1$, V_-^4, $\hat{b}_p V_-^2 \in L^1(\Omega)$,*
4. *$|\{\mathbf{x} \in \Omega : |V(\mathbf{x})| > 2\}| = 0$.*

Then, for all $\gamma \geq 0$,

$$\begin{aligned} \lim_{\lambda\to 0^+} \lim_{\hbar\to 0} \quad & \left(\tfrac{\hbar^3}{\mu\hbar+1}\right)\left\{\tfrac{1}{2}[M_\gamma(\mathbb{D}_V, \Omega, I(\lambda)) + M_\gamma(\mathbb{D}_{-V}, \Omega, I(\lambda))] \right. \\ & \left. -\hbar^{-3}\mathfrak{B}_\gamma(\mu\hbar|B|, \lambda_1, \Omega)\right\} = 0, \end{aligned}$$

where $\lambda_1 = 2|V| + V^2$ and $I(\lambda) = (-\sqrt{1-\lambda}, \sqrt{1-\lambda})$.

Proof. The proof follows from Theorem 6 just as Theorem 5 follows from Theorem 2. As for (55) and (60), we show that (in the notation of (A_1)-(A_3) in §2)

$$\begin{aligned} \liminf_{\hbar\to 0} \quad & \left(\tfrac{\hbar^3}{\mu\hbar+1}\right)\left\{(1/2)\left[N((\mathbb{D}_{V_0}, \Omega, I(\lambda)) + N((\mathbb{D}_{-V_0}, \Omega, I(\lambda))\right]\right. \\ & \left. -\hbar^{-3}\left[\mathfrak{B}(\mu\hbar|B|, -\lambda_1^0, \Omega) + \mathfrak{B}(\mu\hbar|B|, -\lambda_{-1}^0, \Omega)\right]\right\} \geq 0 \end{aligned}$$

where $\lambda_{\pm 1}^0(\lambda) = (\sqrt{1-\lambda} \pm |V_0|)^2 - 1$, and

$$\begin{aligned} \limsup_{\hbar\to 0} \quad & \left(\tfrac{\hbar^3}{\mu\hbar+1}\right)\left\{(1/2)\left[N((\mathbb{D}_{V_0}, \Omega, I(\lambda)) + N((\mathbb{D}_{-V_0}, \Omega, I(\lambda))\right]\right. \\ & \left. -\hbar^{-3}\left[\mathfrak{B}(\mu\hbar|B|, -\lambda_1^0, \Omega) - \mathfrak{B}(\mu\hbar|B|, -\lambda_{-1}^0, \Omega)\right]\right\} \leq 0 \end{aligned}$$

The rest of the proof is similar to that of Theorem 5. □

7. Appendix

7.1. **Approximation of gauges in cubes.** Let $\mathbf{x}_{Q_k} = (x^1_{Q_k}, x^2_{Q_k}, x^3_{Q_k})$ be the center of the cube Q_k. Since $\mathrm{div}(\mathbf{B}) = 0$, we may choose the gauge $\boldsymbol{a} = (a_1, a_2, 0)$

$$\begin{aligned} a_1(\mathbf{x}) :=& \quad -\tfrac{1}{2}\int_{x^2_{Q_k}}^{x_2} B_3(x_1, t, x^3_{Q_k})dt + \int_{x^3_{Q_k}}^{x_3} B_2(x_1, x_2, t)dt, \\ a_2(\mathbf{x}) :=& \quad \tfrac{1}{2}\int_{x^1_{Q_k}}^{x_1} B_3(t, x_2, x^3_{Q_k})dt - \int_{x^3_{Q_k}}^{x_3} B_1(x_1, x_2, t)dt \end{aligned}$$

– see the proof of Lemma 3.1 of Sobolev [23]. Let $\mathbf{B}^o(\mathbf{x}) := \mathbf{B}(\mathbf{x}_{Q_k})$, $\mathbf{x} \in Q_k$ and define the piecewise linear vector potential $\mathring{\mathbf{a}}$ by

$$\begin{aligned} \mathring{a}_1(\mathbf{x}) \quad &:= \quad -\frac{1}{2}B_3(\mathbf{x}_{Q_k})(x_2 - x^2_{Q_k}) + B_2(\mathbf{x}_{Q_k})(x_3 - x^3_{Q_k}), \\ \mathring{a}_2(\mathbf{x}) \quad &:= \quad \frac{1}{2}B_3(\mathbf{x}_{Q_k})(x_1 - x^1_{Q_k}) - B_1(\mathbf{x}_{Q_k})(x_3 - x^3_{Q_k}), \\ \mathring{a}_3 := & \quad 0. \end{aligned}$$

for $\mathbf{x} \in Q_k$. Then $\nabla \times \mathring{\mathbf{a}} = \mathbf{B}^o$ in Q_k and (11) is satisfied.

References

[1] J. Avron, I. Herbst, and B. Simon. Schrödinger operators with magnetic fields. I. General interactions. *Duke Math. J.*, **45**(4), 847–833 (1978).

[2] L. Bugliaro, C. Fefferman, J. Frölich, G. M. Graf, and J. Stubbe. A Lieb-Thirring bound for a magnetic Pauli Hamiltonian. *Commun. Math. Phys.*, **187**, 567-582 (1997).

[3] Y. Colin de Verdiere. L'asymptotique de Weyl pour les bouteilles magnetiques. *Commun. Math. Phys.*, **105**, 327-335 (1986).

[4] B. A. Dubrovin and S. P. Novikov. Ground states in a periodic field. Magnetic Bloch functions and vector bundles. *Soviet Math. Dokl.* **22** (1), 240-244 (1980).

[5] D. E. Edmunds and W. D. Evans. *Spectral Theory and Differential Operators*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1st edition, 1987.

[6] László Erdős. Magnetic Lieb-Thirring inequalities. *Commun. Math. Phys.* **170**, 629-668 (1995).

[7] László Erdős and Jan Philip Solovej. Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields, I. Non-asymptotic Lieb-Thirring estimate. *Duke Math. J.*, **96**(1), 127-173 (1999).

[8] László Erdős and Jan Philip Solovej. Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields, II. Leading order asymptotic estimates. *Commun. Math. Phys.*, **188**, 599-656 (1997).

[9] W. D. Evans and Roger T. Lewis. Eigenvalue estimates in the semi-classical limit for Pauli and Dirac operators with a magnetic field. *Proceedings of the Royal Society of London, A*, **455**, 183-217 (1999).

[10] William Desmond Evans, Roger T. Lewis, Heinz Siedentop, and Jan Philip Solovej. Counting eigenvalues using coherent states with an application to Dirac and Schrödinger operators in the semi-classical limit. *Arkiv för matematik*, **34**(2), 265–283 (1996).

[11] A. Iwatsuka and H. Tamura. Asymptotic distribution of eigenvalues for Pauli operators with non-constant magnetic fields. *Duke J. Math.*, **94**, 535-574 (1998).

[12] Tosio Kato. Remarks on Schrödinger operator with vector potentials. *Integral Equations and Operator Theory*, **1**, 103-113 (1978).

[13] Elliott H. Lieb and Michael Loss. *Analysis*. Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1st edition, 1997.

[14] Elliott H. Lieb, Michael Loss and Jan Philip Solovej. Stability of matter in magnetic fields. *Phys. Rev. Lett.*, **75**, 985-989 (1995).

[15] Elliott H. Lieb, Jan Philip Solovej, and Jakob Yngvason. Asymptotics of heavy atoms in high magnetic fields: II. Semiclassical regions. *Commun. Math. Phys.*, **161**, 77-124 (1994).

[16] Elliott H. Lieb, Jan Philip Solovej, and Jakob Yngvason. Ground states of large quantum dots in magnetic fields. *Phys. Rev. B*, **51**, 10646-10665 (1995).

[17] H. Leinfelder and C. G. Simader. Schrödinger operators with singular magnetic vector potentials. *Math. Z.*, **176**, 1-19 (1981).
[18] G. D. Raikov. Eigenvalue asymptotics for the Pauli operator in strong non-constant magnetic fields. *Manuscript*
[19] G. Rozenblum and M. Solomyak. CLR-estimates for the generators of positivity preserving and positivity dominated semigroups. Vienna, Preprint ESI 447 (1997).
[20] Zhongwei Shen. On moments of negative eigenvalues for the Pauli operator. *J. Differential Equations*, **151**(2), 420-455 (1999).
[21] Barry Simon. Maximal and minimal Schrödinger forms. *J. Operator Theory*, **1**, 37-47 (1979).
[22] A. V. Sobolev. On the Lieb-Thirring estimates for the Pauli operator. *Duke Math. J.*, **82**, 607-635 (1996).
[23] A. V. Sobolev. Quasi-classical asymptotics for the Pauli operator. *Commun. Math. Phys.*, **194**(1), 109-134 (1998).

School of Mathematics, Cardiff University, 23 Senghennydd Road, Cardiff CF2 4YH, UK

E-mail address: BalinskyA@cardiff.ac.uk

School of Mathematics, Cardiff University, 23 Senghennydd Road, Cardiff CF2 4YH, UK

E-mail address: EvansWD@cardiff.ac.uk

Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170, USA

E-mail address: lewis@math.uab.edu